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REGULARITY OF ULTRAFILTERS AND THE CORE MODEL

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ABSTRACT

We show that in the core model every uniform ultrafilter is regular. In addition, we prove that the existence of a nonregular uniform ultrafilter on a singular cardinal implies the existence of an inner model with a measurable cardinal.

§0. Introduction

Regular ultrafilters were introduced because they yield ultrapowers of maximal cardinality. Their basic properties including refinements of the notation of regularity can be found in [3]. Chang and Keisler also formulated various questions concerning the existence of nonregular ultrafilters (see Conjectures 4, 14, 15, 16) in [3]. In this paper we shall show that one cannot prove the existence of a nonregular uniform ultrafilter on an infinite set in ZFC + GCH (if ZF is consistent).

Various partial results in this direction have been known before. Prikry showed in [15] that assuming $V = L$ every uniform ultrafilter on ω_1 is regular. His proof actually showed that assuming $V = L$ every uniform ultrafilter on κ^+ is (κ, κ^+)-regular. Chang analyzed how one could improve this result (see [2]). This was used by Jensen in [8] to show that in L every (γ^+, κ)-regular ultrafilter is (γ, κ)-regular for all regular κ. Especially, in L every uniform ultrafilter on ω_n, $n < \omega$, is regular.

Later, Benda found a simpler proof of Prikry's result. He showed that a weak version of the Kurepa hypothesis for κ^+ implies that every uniform ultrafilter

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on κ^+ is (κ, κ^+)-regular (see [1]). This proof was used by Ketonen to show that $\neg 0^\#$ is sufficient to get the same conclusion (see [12]). Using the core model Jensen weakened the assumption to $\neg L^\mu$ (see [7]). Here "$\neg L^\mu$" is an abbreviation of the statement "there exists no inner model with a measurable cardinal". Moreover, Jensen proved a new result about weakly normal ultrafilters. Using a theorem of Kanamori and Ketonen (see [10]) this result has the following consequence concerning regularity:

(*) Assume $\neg L^\mu$, κ regular and $2^\kappa = \kappa$. Then every uniform ultrafilter on κ is (ω, λ)-regular for all $\lambda < \kappa$.

For uniform ultrafilters on singular cardinals no result in this direction seems to have been known. In this paper we shall show (see Theorem 4.1):

(A) Assume $\neg L^\mu$. Let $\kappa > \omega$ be a singular cardinal. Then every uniform ultrafilter on κ is regular.

For regular cardinals we need a stronger assumption (see Theorem 4.3):

(B) Assume $\neg L^\mu$. Let $\kappa > \omega$ be regular and assume $(\kappa^+)^K = \kappa^+$. Then every uniform ultrafilter on κ is regular.

Especially, we get that in K every uniform ultrafilter on an infinite set is regular. Finally, we use our method to eliminate the assumption $2^\kappa = \kappa$ in (*) (see Theorem 4.5).

Of course, it is a natural question whether one can remove the assumption $(\kappa^+)^K = \kappa^+$ in (B). This problem remains open even if we strengthen $\neg L^\mu$ to $\neg 0^\#$.

In the other direction many results are known. Prikry showed that one can have a uniform ultrafilter on a singular cardinal κ which is even λ-indecomposable for all $\omega < \lambda < \kappa$, if one assumes the consistency of a measurable cardinal (see [14]). This shows that the assumption $\neg L^\mu$ is necessary in (A). Using a huge cardinal Magidor constructed a model in which there is a uniform nonregular ultrafilter on ω_2 (see [13]). Very recently, Foreman, Magidor and Shelah obtained the analogous result for ω_1.

Finally, we should mention a result of Prikry and Silver. They showed that every uniform ultrafilter on a regular κ is λ-decomposable if there is a nonreflecting stationary subset of κ consisting only of ordinals of cofinality λ (see [16]). The proof of this theorem given in [11] was a key to the results in this paper.

The paper is organized as follows. In §1 we prove some preliminary facts about ultrafilters. Moreover, we prove a new result which introduces some of the ideas used later. In §2 we define the crucial combinatorial principles. They are all strengthenings of $\square_\kappa$. We also reduce these principles to some technical statements about "Σ_1-collapsing" structures. The second combinatorial principle was formulated by Jensen after he saw the original version of our proof. There we used a weaker principle. In §3 we prove these principles in the core model K. The main part of the method used in that section is known (e.g. see [17]). §4 contains the main results of this paper which were stated above.

§1. Ultrafilters

In this section we introduce some basic definitions and prove simple lemmas about ultrafilters.

DEFINITION. Let U be an ultrafilter. Then U is *uniform* iff all members of U have the same cardinality.

It suffices to investigate uniform ultrafilters, for any ultrafilter U determines a uniform ultrafilter $\bar{U}$ such that U and $\bar{U}$ have the same structural properties. For our purposes it is also sufficient to consider only ultrafilters on cardinals.

DEFINITION. Let U be an ultrafilter on κ and let λ, τ be cardinals. U is (λ, τ)-*regular* iff there is a sequence $\langle X_\alpha \mid \alpha < \tau \rangle$, $X_\alpha \in U$, such that $\bigcap_{\alpha \in B} X_\alpha = \emptyset$ for all $B \subseteq \tau$, $|B| \geqq \lambda$. U is *regular* iff U is (ω, κ)-regular.

It is easy to see that for an ultrafilter U on κ the following two properties are equivalent.

(a) U is (λ, τ)-regular.

(b) There is a sequence $\langle u_\nu \mid \nu < \kappa \rangle$ such that $u_\nu \subseteq \tau$, $|u_\nu| < \lambda$, and $\forall \alpha < \tau$ $\{\nu \mid \alpha \in u_\nu\} \in U$.

A sequence as in (b) will be called a (λ, τ)-*covering* of U.

LEMMA 1.1. *Let U be an ultrafilter on κ. Let τ be a singular cardinal, λ regular and let U be (λ, ρ)-regular for all $\rho < \tau$. Then U is (λ, τ)-regular.*

PROOF. Let $\bar{\tau} = \mathrm{cf}(\tau)$. Let $\langle \tau_\delta \mid \delta < \bar{\tau} \rangle$ be a normal sequence of cardinals such that $\tau_0 = 0$ and $\tau = \sup\{\tau_\delta \mid \delta < \bar{\tau}\}$. Let $\langle u_\nu^\delta \mid \nu < \kappa \rangle$ be a (λ, τ_0)-covering of U and $\langle \bar{u}_\nu \mid \nu < \kappa \rangle$ be a $(\lambda, \bar{\tau})$-covering of U. For $\nu < \kappa$ set

$$u_\nu = \bigcup \{u_\nu^{\delta+1} \mid \delta \in \bar{u}_\nu\}.$$

Since λ is regular we have $|u_\nu| < \lambda$. Now let $\alpha < \tau$. Choose δ such that $\tau_\delta \leqq \alpha < \tau_{\delta+1}$. Then

$$\{\nu < \kappa \mid \alpha \in u_\nu\} \supseteq \{\nu < \kappa \mid \delta \in \bar{u}_\nu\} \cap \{\nu < \kappa \mid \alpha \in u_\nu^{\delta+1}\} \in U.$$

So $\langle u_\nu \mid \nu < \kappa \rangle$ is a (λ, τ)-covering of U. qed

DEFINITION. Let U be an ultrafilter and $f: X \to \lambda$, $X \in U$. Then f is a *λ-decomposition* of U iff for all $\alpha < \lambda$ $\{\nu \in X \mid f(\nu) \geqq \alpha\} \in U$.

For singular λ this notation is slightly misleading but it is good enough for our purposes.

The following translation of regularity is useful for us. We do not state the most general version. If $f, g: X \to \mathrm{On}$ then $f \leqq g$ means that $f(x) \leqq g(x)$ for all $x \in X$.

LEMMA 1.2. *Let U be an ultrafilter on κ and let $\tau > \omega$ be regular. Then the following properties are equivalent*:

(a) *U is (ω, τ)-regular.*

(b) *There are $B \subseteq \mathrm{On}$, $\mathrm{otp}(B) = \tau$, B closed in $\sup B$, $X \in U$ and λ-decompositions $f_\lambda: X \to \lambda$ for $\lambda \in B$ such that $f_\lambda \leqq f_\mu$ for all $\lambda, \mu \in B$, $\lambda \leqq \mu$.*

PROOF. (a)$\to$(b) Let $\langle u_\gamma \mid \gamma < \kappa \rangle$ be a (ω, τ)-covering of U. Set $B = \tau - \{0\}$. For $\lambda \in B$ define $f_\lambda: \kappa \to \lambda$ by $f_\lambda(\gamma) = \sup(u_\gamma \cap \lambda)$. These satisfy (b).

(b)$\to$(a) Let B, X, f_λ be given. For $\eta \in B$ let

$$X_\eta = \{\gamma \in X \mid \forall \delta \in B - (\eta + 1) f_\delta(\gamma) \geqq \eta\}.$$

By assumption we have that

$$X_\eta = \{\gamma \in X \mid f_\mu(\gamma) \geqq \eta\}$$

where $\mu = \min(B - (\eta + 1))$. So $X_\eta \in U$, since f_μ is a μ-decomposition of U. We show that $\langle X_\eta \mid \eta \in B \rangle$ gives the (ω, τ)-regularity of U. So let $\langle \eta(n) \mid n < \omega \rangle$ be a monotone sequence such that $\eta(n) \in B$ and let $\eta = \sup\{\eta(n) \mid n < \omega\}$. Then $\eta \in B$ since B is closed in $\sup B$ and $\mathrm{cf}(\tau) > \omega$. We have to show that $\bigcap_{n<\omega} X_{\eta(n)} = \varnothing$. Assume that this is not the case. So let $\alpha \in \bigcap_{n<\omega} X_{\eta(n)}$. Then $f_\eta(\alpha) \geqq \eta(n)$ for all $n < \omega$, so $f_\eta(\alpha) \geqq \eta$. But this contradicts the property $f_\eta: X \to \eta$. qed

If U is an ultrafilter on κ and $f: \kappa \to \lambda$, then let

$$f^*(U) = \{X \subseteq \lambda \mid f^{-1}{}'' X \in U\}.$$

Clearly, $f^*(U)$ is an ultrafilter on λ. Note that U is (μ, ρ)-regular, if $f^*(U)$ is (μ, ρ)-regular. Kanamori showed in [10] that for any uniform nonregular ultrafilter on a regular $\kappa > \omega$ there is some $f: \kappa \to \kappa$ such that $f^*(U) \supseteq \mathscr{C}_\kappa$, where $\mathscr{C}_\kappa$ is the filter generated by the club subsets of κ. We also need a version of Kanamori's result for ultrafilters on singular cardinals. If U is an ultrafilter, $f: X \to \mathrm{On}$, $g: Y \to \mathrm{On}$, $X, Y \in U$, then $f <_U g$ denotes that $\{x \in X \cap Y \mid f(x) < g(x)\} \in U$.

Lemma 1.3. *Let κ be a singular cardinal. Let $\langle \kappa_\delta \mid \delta < \rho \rangle$, $\rho < \kappa$, be a sequence of limit cardinals such that:*

(i) $\kappa > \kappa_\delta > \sup\{\kappa_\xi \mid \xi < \delta\}$.

(ii) $\langle \mathrm{cf}(\kappa_\delta) \mid \delta < \rho \rangle$ *is weakly monotone.*

(iii) $\kappa = \sup\{\mathrm{cf}(\kappa_\delta) \mid \delta < \rho\}$.

Let U be a uniform ultrafilter on κ which is not regular. Then there is some $f: \kappa \to \kappa$ such that

$$f^*(U) \supseteq \{C \subseteq \kappa \mid \forall \delta < \rho\, C \cap \kappa_\delta \text{ club in } \kappa_\delta\}.$$

Proof. Let F be the set of all $f: \kappa \to \kappa$ which satisfy

(1) f is weakly monotone,

(2) $\exists \delta < \rho\ \forall \eta \geqq \delta\ f''\kappa_\eta$ unbounded in κ_η.

It suffices to show

(*) F has a least element mod U.

To see this, assume that f is such a function. Then $\mathrm{id} \restriction \kappa$ is the least element of F mod $f^*(U)$. Now let $C \subseteq \kappa$ such that $\forall \delta < \rho\, C \cap \kappa_\delta$ is club in κ_δ. Define $g: \kappa \to \kappa$ by $g(\alpha) = \sup(C \cap \alpha)$. Then $g \in F$ and $g \leqq \mathrm{id} \restriction \kappa$. So $A = \{\alpha < \kappa \mid g(\alpha) = \alpha\} \in f^*(U)$. But $A \subseteq C \cup B$, where $B = (\min C + 1) \cup \{\kappa_\delta \mid \delta < \rho\}$. So $C \in f^*(U)$, since $|B| < \kappa$ and $f^*(U)$ is uniform.

So we have to prove (*). Assume that (*) is false. We define a sequence $\langle f_\alpha \mid \alpha < \kappa \rangle$ such that $f_\alpha \in F$ and

(a) $f_{\alpha+1} <_U f_\alpha$,

(b) $f_\beta \leqq f_\alpha$ for all $\alpha < \beta < \kappa$,

(c) f_α satisfies the condition (2) for the least $\delta < \rho$ such that $\mathrm{cf}(\kappa_\delta) > \alpha$.

The definition of this sequence is done by recursion. Successor steps and the initial case are obvious. If $\lambda < \kappa$ is a limit ordinal we just set

$$f_\lambda(\gamma) = \min\{f_\alpha(\gamma) \mid \alpha < \lambda\}.$$

Finally, we then set $X_\alpha = \{\gamma < \kappa \mid f_{\alpha+1}(\gamma) < f_\alpha(\gamma)\}$. Then $\langle X_\alpha \mid \alpha < \kappa \rangle$ shows that U is regular. This contradicts our assumption. qed

The proof of the next result is a good introduction to the method used in this paper to get regularity properties of ultrafilters. But first we need a definition.

DEFINITION. Let $\kappa > \omega$ be regular. Then $\square_\kappa^-$ denotes the following principle:

There is a sequence $\langle C_\gamma \mid \gamma < \kappa, \lim(\gamma) \rangle$ such that:

(a) $C_\gamma \subseteq \gamma$ is closed in γ,

(b) $\mathrm{cf}(\gamma) > \omega \to \sup C_\gamma = \gamma$,

(c) $\lambda \in C_\gamma \to C_\lambda = C_\gamma \cap \lambda$,

(d) there is no unbounded $C \subseteq \kappa$ such that $C \cap \lambda = C_\lambda$ for all $\lambda \in C$.

So for $\kappa = \lambda^+$ $\square_\kappa^-$ is a weaker version of the more familiar principle $\square_\lambda$. We hope that the different indices are not confusing. Clearly, $\square_\kappa^-$ cannot hold if κ is weakly compact. In [9] Jensen has shown that $\square_\kappa^-$ holds in L for all regular $\kappa > \omega$ which are not weakly compact. Note that $\square_{\omega_1}^-$ is provable in ZFC.

THEOREM 1.4. *Let $\kappa > \omega$ be regular and assume that $\square_\kappa^-$ holds. Then every uniform ultrafilter U on κ is (ω, τ)-regular for every $\tau < \kappa$.*

PROOF. For $\kappa = \omega_1$ this is a classical result. So let $\kappa > \omega_1$ and let U be given. It suffices to prove the claim for all regular τ such that $\omega < \tau < \kappa$. So let such a τ be given and let $\langle C_\gamma \mid \gamma < \kappa, \lim(\gamma) \rangle$ be a $\square_\kappa^-$-sequence. Set $D = \{\gamma < \kappa \mid \sup C_\gamma = \gamma\}$. We first define a sequence $\langle g_\alpha \mid \alpha < \kappa \rangle$ such that $g_\alpha : D \cap (\alpha + 1) \to \alpha$ and the following three properties hold for all $\alpha < \kappa$:

(i) g_α is regressive.

(ii) Let $\gamma, \lambda \in D$, $\gamma < \lambda \leqq \alpha$, and $\gamma \notin C_\lambda$. Then

$$(C_\gamma - g_\alpha(\gamma)) \cap (C_\lambda - g_\alpha(\lambda)) = \varnothing.$$

(iii) Let $\gamma, \lambda \in D \cap (\alpha + 1)$ and $\gamma \in C_\lambda$. Then $g_\alpha(\gamma) \leqq g_\alpha(\lambda)$.

The definition is done by recursion. Set $g_0 = \varnothing$ and $g_{\alpha+1} = g_\alpha$. Now let $\alpha < \kappa$ be a limit ordinal. If $\sup C_\alpha < \alpha$ (hence $\mathrm{cf}(\alpha) = \omega$) choose a monotone ω-sequence H such that $\sup H = \alpha$ and $\sup C_\alpha < \min H$. If $\sup C_\alpha = \alpha$, set $H = \varnothing$. Now first let $\gamma \in (D \cap \alpha) - C_\alpha$. Set $\mu = \sup((C_\alpha \cup H) \cap \gamma)$, so $\mu < \gamma$. Let ρ be the successor of μ in $C_\alpha \cup H$. Then set $g_\alpha(\gamma) = \max\{g_\rho(\gamma), \mu\}$. If $\gamma \in (C_\alpha \cup \{\alpha\}) \cap D$ set $g_\alpha(\gamma) = 0$. It is easy to see that g_α satisfies (i)–(iii).

For $\gamma \in D$ we now define $f_\gamma : \kappa - \gamma \to \gamma$ by $f_\gamma(\alpha) = g_\alpha(\gamma)$. By (iii) we get

(1) Let $\gamma, \lambda \in D$, $\gamma \in C_\lambda$. Then $f_\gamma(\alpha) \leqq f_\lambda(\alpha)$ for all $\lambda \leqq \alpha < \kappa$.

We now show

(2) Set $A = \{\gamma \in D \mid f_\gamma$ is not a γ-decomposition of $U\}$. Then A is not stationary in κ.

PROOF. By definition of A there is a regressive function $h: A \to \kappa$ such that

$$X_\gamma = \{\alpha \in \kappa - \gamma \mid f_\gamma(\alpha) \leqq h(\gamma)\} \in U \qquad \text{for } \gamma \in A.$$

Assume that A were stationary. Define $g: A \to \kappa$ such that $g(\gamma) \in C_\gamma - h(\gamma)$. Then by Fodor g is constant on some stationary $E \subseteq A$. But then we have

(*) Let $\gamma, \lambda \in E$, $\gamma < \lambda$. Then $\gamma \in C_\lambda$.

To see this choose $\alpha \in X_\gamma \cap X_\lambda (\in U)$. Then

$$g(\gamma) \in (C_\gamma - g_\alpha(\gamma)) \cap (C_\lambda - g_\alpha(\lambda)).$$

So $\gamma \in C_\lambda$ follows from (ii). But (*) contradicts the definition of a $\square_\kappa^-$-sequence.
qed (2)

Now choose some club $C \subseteq \kappa$ such that $C \cap A = \emptyset$ and let γ be a limit point of C such that $\mathrm{cf}(\gamma) = \tau$. Then $\sup C_\gamma = \gamma$ since $\tau > \omega$. Let $B \subseteq C \cap C_\gamma$ be club in γ and $\mathrm{otp}(B) = \tau$. Set $X = \kappa - \gamma$ and for $\delta \in B$ set $\bar{f}_\delta = f_\delta \upharpoonright X$. Then $\langle \bar{f}_d \mid \delta \in B \rangle$ satisfies the condition (b) in Lemma 1.2. Hence U is (ω, τ)-regular.
qed

§2. □-principles

In this section we introduce the combinatorial principles we need for our results.

We start with some definitions and notations. Set

$$\mathscr{S} = \{s = \langle \nu_s, a_s \rangle \mid \lim(\nu_s) \text{ and } a_s \subseteq J_\eta, \text{ where } \omega\eta = \nu_s\}.$$

For $s \in \mathscr{S}$ set

$$J_s = \langle J_\eta^a, a \rangle, \qquad \text{where } a = a_s, \quad \omega\eta = \nu_s.$$

Let h_s be the canonical Σ_1-Skolem function of J_s. If $\bar{s}, s \in \mathscr{S}$ we write $f: \bar{s} \Rightarrow s$ for $f: J_{\bar{s}} \to_{\Sigma_1} J_s$. The notation $f \Rightarrow s$ denotes that $f: \bar{s} \Rightarrow s$ for some $\bar{s}$. Set

$$F = \{f \mid f \Rightarrow s \text{ for some } s \in \mathscr{S}\}.$$

Now let $s \in \mathscr{S}$ and $\lambda \leqq \nu_s$, $\lim(\lambda)$. Then set $s \mid \lambda = \langle \lambda, a \cap J_\eta^a \rangle$, where $a = a_s$, $\omega\eta = \lambda$.

For $f: \bar{s} \Rightarrow s$ set

$$\lambda(f) = \sup(\mathrm{On} \cap \mathrm{rng}(f)) \quad \text{and} \quad \beta(f) = \sup\{\beta \leqq \nu_s \mid f \upharpoonright \beta = \mathrm{id} \upharpoonright \beta\}.$$

So if $f \neq \mathrm{id}$, $\beta(f)$ is the critical point of f.

We now define some special elements of F as follows. Let $s \in \mathcal{S}$ and $\beta \leqq \nu_s$. Let

$$X = \text{the } \Sigma_1\text{-Skolem hull of } \beta \text{ in } J_s.$$

Let $f: M \xrightarrow{\sim} X$, where M is transitive. Then there is exactly one $\bar{s} \in \mathcal{S}$ such that $M = J_{\bar{s}}$. We also have that $f: \bar{s} \Rightarrow s$. Set $f_{(\beta,s)} = f$.

We now define a sequence $\langle C_s \mid s \in \mathcal{S} \rangle$. So let $s \in \mathcal{S}$. For $\delta < \nu_s$ set

$$W_\delta = W_\delta^s = \mathrm{On} \cap h_s''(\omega \times \{\delta\}).$$

For $\eta < \nu_s$ set

$$\gamma(\eta) = \gamma_s(\eta) = \sup\{\sup W_\delta \mid \delta < \eta\}.$$

Then set

$$I_s = \{\eta \leqq \nu_s \mid \forall \bar{\eta} < \eta \gamma(\bar{\eta}) < \gamma(\eta) < \nu_s\}.$$

Finally we set

$$C_s = \{\gamma(\eta) \mid \eta \in I_s\}.$$

Note that C_s is uniformly definable in J_s. Clearly, $C_s \subseteq \{\gamma < \nu_s \mid \lim(\gamma)\}$, since $W_\delta \prec_{\Sigma_1} \nu_s$. The next theorem contains the main properties of the sequence $\langle C_s \mid s \in \mathcal{S} \rangle$.

THEOREM 2.1. *Let $s \in \mathcal{S}$. Then:*

(a) *$C_s \subseteq \nu_s$ is closed in ν_s.*

(b) $\lambda \in C_s \rightarrow C_{s|\lambda} = \lambda \cap C_s$.

(c) $\mathrm{cf}(\nu_s) > \omega \rightarrow \sup C_s = \nu_s$.

(d) *Let $\alpha < \nu_s$ such that $\lambda(f_{(\alpha,s)}) = \nu_s$. Set*

$$\bar{\alpha} = \min\{\bar{\alpha} \geqq \alpha \mid \bar{\alpha} \textit{ primitive recursively closed}\}.$$

Then $\mathrm{otp}\, C_s \leqq \bar{\alpha}$.

(e) *Let $f: \bar{s} \Rightarrow s$. Then $f: \langle J_{\bar{s}}, C_{\bar{s}} \rangle \rightarrow_{\Sigma_0} \langle J_s, C_s \rangle$ and $\langle J_s, C_s \rangle$ is amenable.*

PROOF. (a) is obvious.

(b) Let $\lambda \in C_s$. So $\lambda = \gamma_s(\eta)$ for some $\eta \in I_s$. By basic properties of the canonical Σ_1-Skolem functions we get that $W_\delta^s = W_\delta^{s|\lambda}$ for all $\delta < \eta$ and $I_{s|\lambda} = \eta \cap I_s$. So $\gamma_s(\bar{\eta}) = \gamma_{s|\lambda}(\bar{\eta})$ for all $\bar{\eta} < \eta$. Hence $C_{s|\lambda} = \lambda \cap C_s$.

(c) Set $\nu = \nu_s$ and $W_\delta = W_\delta^s$. Let $\mathrm{cf}(\nu) > \omega$. We have to show that $\sup C_s = \nu$. It suffices to show that $I_s \neq \varnothing$ and I_s has no maximal element. So let $\bar{\eta} \in I_s \cup \{0\}$. We are looking for some $\eta \in I_s$ such that $\eta > \bar{\eta}$. Now set

$\delta = \min\{\delta \geqq \bar{\eta} \mid \sup W_\delta > \gamma(\bar{\eta})\}$. Then $\sup W_\delta < \nu$, since $|W_\delta| \leqq \omega$ and $\mathrm{cf}(\nu) > \omega$. But then by definition $\delta + 1 \in I_s$.

(d) Let $G : [\mathrm{On}]^{<\omega} \to \mathrm{On}$ be the canonical primitive recursive bijection. Set $\alpha' = \sup G''[\alpha]^{<\omega}$. Then we have

$$\mathrm{On} \cap \mathrm{rng}(f_{(\alpha,s)}) \subseteq \bigcup \{W_\delta \mid \delta < \alpha'\},$$

hence $I_s \subseteq \alpha'$ and $\mathrm{otp}\, C_s \leqq \alpha'$.

(e) By the uniform definability of $\langle C_s \mid s \in \mathscr{S} \rangle$ we get by (b) that $\langle J_s, C_s \rangle$ is amenable and that $f(C_{s|\bar{\lambda}}) = C_{s|f(\bar{\lambda})}$ for all $\bar{\lambda}$ such that $\omega\bar{\lambda} < \nu_s$. So it suffices to show that $F''C_s \subseteq C_s$. So let $\bar{\lambda} \in C_s$, say $\bar{\lambda} = \gamma_s(\bar{\eta})$, $\bar{\eta} \in I_s$. Set $\eta = f(\bar{\eta})$. We show that $\eta \in I_s$ and $f(\bar{\lambda}) = \lambda$. Set $\lambda = f(\bar{\lambda})$. Since f is Σ_1-elementary the following Π_1-statement holds in J_s:

(1) $\forall \delta < \eta\ \forall i \in \omega\, (\langle i, \delta \rangle \in \mathrm{dom}\, h_s \to h_s(i, \delta) < \lambda)$.

But $f \upharpoonright J_{s|\bar{\lambda}}$ is an elementary map of $J_{s|\bar{\lambda}}$ into $J_{s|\lambda}$. Hence

(2) $\forall \rho < \lambda\ \exists i\ \exists \delta < \eta\, (\langle i, \delta \rangle \in \mathrm{dom}\, h_{s|\lambda}$ and $h_{s|\lambda}(i, \delta) > \rho)$.

By (1) we can replace $h_{s|\lambda}$ by h_s in (2). So $\eta \in I_s$ and $\lambda = \gamma_s(\eta)$. qed

We now formulate the first crucial combinatorical principle.

Definition. Let $\kappa > \omega$ be regular. Then $\square_\kappa^+$ denotes the following combinatorical principle.

There are $\langle S_\alpha \mid \alpha \leqq \kappa \rangle$, $\langle C_\nu \mid \nu \in S_\alpha \rangle$, $\langle A_\nu \mid \nu \in S_\kappa \rangle$ and $G_\alpha : \{\nu \in S_\kappa \mid \alpha \in A_\nu\} \to S_\alpha$ for $\alpha < \kappa$ such that the following properties holds:

(E0) (a) $S_\alpha \subseteq \alpha^+$ is closed in α^+ for $\alpha \leqq \kappa$.
(b) S_κ is unbounded in κ^+.
(c) $\sup S_\alpha < \kappa$ for $\alpha < \kappa$.
(d) $A_\nu \subseteq \kappa$ is club in κ for $\nu \in S_\kappa$.

(E1) For $\alpha \leqq \kappa$, $\nu \in S_\alpha$, we have
(a) $C_\nu \subseteq S_\alpha \cap \nu$ is closed in ν.
(b) $\lambda \in C_\nu \to C_\lambda = \lambda \cap C_\nu$.
(c) $\mathrm{cf}(\nu) > \omega \to \sup C_\nu = \nu$.
(d) $\mathrm{otp}\, C_\nu \leqq \alpha$.

(E2) Let $\nu, \tau \in S_\kappa$, $\nu < \tau$. Then there is some $\eta < \kappa$ such that $G_\alpha(\nu) < G_\alpha(\tau)$ for all $\alpha \in (A_\nu \cap A_\tau) - \eta$.
(b) Let $\nu \in S_\kappa$, $\alpha \in A_\nu$. Set $\eta = G_\alpha(\nu)$ and

$$\lambda = \sup\{\lambda \in C_\nu \cup \{\nu\} \mid \mathrm{otp}(C_\nu \cap \lambda) \leqq \alpha\}.$$

Then G_α maps $C_\nu \cap \lambda$ order preservingly onto C_η.

So this principle asserts the existence of a $\square_\kappa$-sequence which projects to partial $\square_\alpha$-sequences. It may be helpful for the reader to draw a picture.

Later we shall show that $\square_\kappa^+$ holds in K for all regular $\kappa > \omega$. We think that it is convenient to divide the proof of this into two parts. We now present the first part which can be stated in a general way without mentioning the special inner model K. For this we introduce the following rather technical definition. Let $\mathscr{S}$ and $\langle C_s \mid s \in \mathscr{S} \rangle$ be as above.

DEFINITION. Let $\kappa > \omega$ be regular. Then $(\mathrm{B})_\kappa$ denotes the following principle.

There are $S \subseteq \kappa^+$ and $\langle s_\nu \mid \nu \in S \rangle$ such that $s_\nu \in \mathscr{S}$ and the following properties hold.

(B0) Let $\nu \in S$, $s = s_\nu$. Then there is some $\alpha > \omega$ such that $J_s \models \nu = \alpha^+$ and $f_{(\alpha,s)} = \mathrm{id}$.

Set $\alpha = \alpha_\nu$ and $S_\alpha = \{\nu \in S \mid \alpha = \alpha_\nu\}$.

(B1) (a) $S_\alpha \subseteq \alpha^+$ is closed for $\alpha \leqq \kappa$.

(b) S_κ is unbounded in κ^+.

(c) $\sup S_\alpha < \kappa$ for $\alpha < \kappa$.

(B2) Let $\nu \in S_\alpha$, $s = s_\nu$, $\lambda \in C_s$, $f = f_{(\alpha, s \mid \lambda)}$. Let $f: \bar{s} \Rightarrow s \mid \lambda$ and set $\eta = \sup\{\rho \leqq \nu_{\bar{s}} \mid f(\rho) \leqq \nu\}$. Then $\eta \in S_\alpha$ and $\bar{s} = s_\eta$.

(B3) Let $\nu \in S_\kappa$, $s = s_\nu$, $f = f_{(\alpha,s)}$. Assume that $\mathrm{rng}(f) \cap \kappa = \alpha$ and let $f: \bar{s} \Rightarrow s$, $\bar{\nu} = f^{-1}{}''\nu$. Then $\bar{\nu} \in S_\alpha$ and $\bar{s} = s_{\bar{\nu}}$.

(B4) Let $\nu, f, \alpha, \bar{\nu}, \bar{s}$ be as in (B3). Further let $\tau \in S_\kappa \cap \nu$ and $f(\bar{\tau}) = \tau$. Then $\bar{\tau} \in S_\alpha$, $s_{\bar{\tau}} \in J_s$ and $f(s_{\bar{\tau}}) = s_\tau$.

LEMMA 2.2. *Let $\kappa > \omega$ be regular and assume that* $(\mathrm{B})_\kappa$ *holds. Then* $\square_\kappa^+$ *holds.*

PROOF. Let $(\mathrm{B})_\kappa$ be given by S, S_α, s_ν. We have to define the different components of $\square_\kappa^+$. For $\alpha \leqq \kappa$ we take the same S_α. So (E0)(a), (b), (c) are satisfied. We now define the C_ν's. So let $\nu \in S_\alpha$ and $s = s_\nu$. Let $\lambda \in C_s$ and $f = f_{(\alpha, s\mid\lambda)}$, $f: \bar{s} \Rightarrow s \mid \lambda$. Then set

$$\lambda^{(\nu)} = \sup\{\rho \leqq \nu_{\bar{s}} \mid f(\rho) \leqq \nu\}.$$

Using this notation we set

$$C_\nu = \{\lambda^{(\nu)} \mid \lambda \in C_s\}.$$

We have to show that (E1) is satisfied. So let $\lambda \in C_s$ and f be as above. By

(B2) we know that $\lambda^{(\nu)}$ is the cardinal successor of α in J_s. Since $f(\alpha)=\alpha$ we get that $f\upharpoonright\lambda^{(\nu)}=\mathrm{id}\upharpoonright\lambda^{(\nu)}$. Hence

$$\lambda^{(\nu)}=\mathrm{rng}(f)\cap\nu=h_{s|\lambda}(\alpha)\cap\nu.$$

Since ν is regular in J_s it follows that $\lambda^{(\nu)}<\nu$. We also get that C_ν is closed in ν since the sequence $\langle\lambda^{(\nu)}\mid\lambda\in C_s\rangle$ is continuous. Now we use the fact that $f_{(\alpha,s)}=\mathrm{id}_s$. Hence by 2.1(d) $\mathrm{otp}(C_s)\leqq\alpha$, since α is primitive recursively closed. Further we get

$$\sup C_s=\nu_s\rightarrow\sup C_\nu=\nu.$$

A similar argument yields that $\mathrm{cf}(\nu_s)=\mathrm{cf}(\nu)$. We still have to show that (E1)(b) is satisfied. So let $\eta\in C_\nu$. Then $\eta=\lambda^{(\nu)}$ for some $\lambda\in C_s$, $s=s_\nu$. Further let $f=f_{(\alpha,s|\lambda)}$. Then (B2) says that $f\colon s_\eta\Rightarrow s\mid\lambda$. Set $\bar{s}=s_\eta$. By Lemma 2.1 we know that $C_{s|\lambda}=C_s\cap\lambda$ and $f''C_{\bar{s}}=C_s\cap\lambda$. But then using $f\upharpoonright\eta=\mathrm{id}\upharpoonright\eta$ it is easy to see that $\bar{\lambda}^{(\eta)}=\tilde{\lambda}^{(\nu)}$ whenever $\bar{\lambda}\in C_{\bar{s}}$ and $\tilde{\lambda}=f(\bar{\lambda})$. This implies that $C_\nu\cap\eta=C_\eta$.

We now define $\langle A_\nu\mid\nu\in S_\kappa\rangle$. So let $\nu\in S_\kappa$ and set $s=s_\nu$, $f_\alpha=f_{(\alpha,s)}$. We set

$$A_\nu=\{\alpha<\kappa\mid\kappa\in\mathrm{rng}\,f_\alpha,\ \kappa\cap\mathrm{rng}\,f_\alpha=\alpha,\ f_\alpha\text{ is elementary}\}.$$

Obviously, A_ν is club in κ. To define G_α we apply (B3). So let $\nu\in S_\kappa$, $\alpha\in A_\nu$, and set $s=s_\nu$, $f=f_{(\alpha,s)}$. We then set $G_\alpha(\nu)=f^{-1}{}''\nu$. By (B3) we know that $G_\alpha(\nu)\in S_\alpha$. We have to show that (E2)(a) holds. So let $\nu,\tau\in S_\kappa$, $\nu<\tau$ and set $s=s_\tau$. Since $f_{(\kappa,s)}=\mathrm{id}_s$ there is some $\eta<\kappa$ such that $\nu\in\mathrm{rng}\,f_{(\alpha,s)}$ for all $\alpha\in A_\tau-\eta$. Now let $\alpha\in A_\tau-\eta$, $f=f_{(\alpha,s)}$ and $f(\bar{\nu})=\nu$. Set $\bar{s}=s_{\bar{\nu}}$, $s'=s_\nu$, $\bar{f}=f\,|\,J_{\bar{s}}$. Then (B4) implies that $\bar{f}\colon\bar{s}\Rightarrow s'$. Since $\kappa\cap\mathrm{rng}\,\bar{f}=\alpha$, $\bar{f}\upharpoonright\alpha=\mathrm{id}\upharpoonright\alpha$ and $f_{(\alpha,s')}=\mathrm{id}_{s'}$ it is easy to see that $\bar{f}=f_{(\alpha,s')}$. Hence $\bar{\nu}=G_\alpha(\nu)<G_\alpha(\tau)$.

Finally we have to show (E2)(b). So let $\nu\in S_\kappa$, $\alpha\in A_\nu$, $\eta=G_\alpha(\nu)$, $s=s_\nu$, $\bar{s}=s_\eta$, $f=f_{(\alpha,s)}$. Hence $f\colon\bar{s}\Rightarrow s$. Since $f\upharpoonright\alpha=\alpha$ and $\mathrm{otp}\,C_s\leqq\alpha$ we know by Theorem 2.1(e) that $f''C_{\bar{s}}$ is an initial segment of C_s. Since f is elementary we have $f(\mathrm{otp}\,C_{\bar{s}})=\mathrm{otp}\,C_s$. So it suffices to show the following statement:

$$\bar{\lambda}\in C_{\bar{s}},\quad\lambda=f(\bar{\lambda})\rightarrow\lambda^{(\nu)}=f(\bar{\lambda}^{(\eta)}).$$

Since then by the argument used for (E2)(a) it follows that $\bar{\lambda}^{(\eta)}=G_\alpha(\lambda^{(\nu)})$, since $\bar{\lambda}^{(\eta)}\in S_\alpha\cap\eta$. Now $f(h_{\bar{s}|\bar{\lambda}}(\alpha))=h_{s|\lambda}(\kappa)$ and $\eta=f^{-1}{}''\nu$. Hence $\lambda^{(\nu)}=f(\bar{\lambda}^{(\eta)})$.

qed

For inaccessible κ $\square_\kappa^+$ will not be sufficient for our results. So we state here some additional information contained in the proof of the last lemma.

LEMMA 2.3. *Let the* $\square_\kappa^+$*-sequence be constructed from* $(\mathrm{B})_\kappa$ *as in the proof above. Then we have:*

There are $\pi_\alpha^\nu : J_{\bar s} \to_{\Sigma_1} J_s$ *for* $\nu \in S_\kappa$, $\alpha \in A_\nu$, $\bar\nu = G_\alpha(\nu)$, $\bar s = s_{\bar\nu}$, $s = s_\nu$ *with the properties:*

(1) $\pi_\alpha^\nu \restriction \alpha = \mathrm{id} \restriction \alpha$, $\pi_\alpha^\nu(\alpha) = \kappa$.

(2) *Let* $\tau \in S_\kappa \cap \nu$, $\tau = \pi_\alpha^\nu(\bar\tau)$, $s' = s_\tau$. *Then* $\pi_\alpha^\tau = \pi_\alpha^\nu \restriction J_{\bar s'}$.

PROOF. For $\nu \in S_\kappa$, $\alpha \in A_\nu$, $s = s_\nu$ we set $\pi_\alpha^\nu = f_{(\alpha,s)}$. The properties (1), (2) were implicitly proved above. qed

We now introduce for arbitrary cardinals $\kappa > \omega$ a similar principle $\square_\kappa^{++}$. We shall only use this principle for singular κ.

DEFINITION. Let $\kappa > \omega$ be a cardinal. Then $\square_\kappa^{++}$ denotes the following principle:

There are $\langle S_\alpha \mid \alpha \leqq \kappa \rangle$, $\langle C_\nu \mid \nu \in S_\alpha \rangle$, $\langle A_\nu \mid \nu \in S_\kappa \rangle$,
$G_\alpha : \{\nu \in S_\kappa \mid \alpha \in A_\nu\} \to S_\alpha$ with the properties:

(E0)(a) (b) as earlier.

(c) $\alpha < \kappa \to \sup S_\alpha < \alpha^+$.

(d) Let $\nu \in S_\kappa$. Then there is some $\rho < \kappa$ such that $A_\nu \cap \lambda^+$ is club in λ^+ for all $\rho \leqq \lambda < \kappa$.

(E1) and (E2) (a) as earlier.

(E2)(b) Let $\nu \in S_\kappa$, $\alpha \in A_\nu$. Set $\eta = G_\alpha(\nu)$ and

$$\bar\lambda = \sup\{\lambda \in C_\nu \cup \{\nu\} \mid \mathrm{otp}(C_\nu \cap \lambda) \leqq \alpha\}.$$

Then G_α maps $C_\nu \cap \bar\lambda$ order preservingly onto a final segment of C_η.

The weakening of (E2)(b) is motivated by the fact that we can only prove this version in suitable inner models. We now formulate an analogue to $(\mathrm{B})_\kappa$.

DEFINITION. Let $\kappa > \omega$ be a cardinal. Then $(\bar{\mathrm{B}})_\kappa$ denotes the following principle:

There are $S \subseteq \kappa^+$ and $\langle s_\nu \mid \nu \in S \rangle$ such that $s_\nu \in \mathscr{S}$ and the following properties hold:

($\bar{\mathrm{B}}$0) Let $\nu \in S$ and $s = s_\nu$. Then there is $\alpha > \omega$ such that $J_s \models \nu = \alpha^+$ and $f_{(\alpha,s)} = \mathrm{id}_s$. Moreover, if $\alpha = \kappa$, then $J_s \models$ "H_κ is a set and $\forall \lambda < \kappa$ $2^\lambda = \lambda^+$".

Set $\alpha = \alpha_\nu$ and $S_\alpha = \{\nu \in S \mid \alpha = \alpha_\nu\}$.

($\bar{\mathrm{B}}$1)(a) $S_\alpha \subseteq \alpha^+$ is closed in α^+.

(b) S_κ is unbounded in κ^+.

(c) $\sup S_\alpha < \alpha^+$ for $\alpha < \kappa$.

($\bar{B}$2) There is $\langle \xi_\nu \mid \nu \in S \rangle$ with the properties:
Let $\nu \in S$, $s = s_\nu$, $\alpha = \alpha_\nu$. Then
(a) $\xi_\nu \leqq \nu_s$; $\mathrm{cf}(\nu) > \omega \to \xi_\nu < \nu_s$.
(b) Let $\lambda \in C_s$, $\lambda > \xi_\nu$, $f = f_{(\alpha, s \mid \lambda)}$, $f: \bar{s} \Rightarrow s \mid \lambda$. Set

$$\eta = \sup\{\xi \leqq \nu_s \mid f(\xi) \leqq \nu\}.$$

Then $\eta \in S_\alpha$, $\bar{s} = s_\eta$ and $f(\xi_\eta) = \xi_\nu$.

($\bar{B}$3) There is $\langle \delta_\nu \mid \nu \in S_\kappa \rangle$, $\delta_\nu < \kappa$, such that the following property holds:
Let $\nu \in S_\kappa$, $s = s_\nu$, $\delta_\nu \leqq \alpha < \kappa$, α no cardinal, $f = f_{(\alpha,s)}$, $f: \bar{s} \Rightarrow s$, $\alpha = \beta(f)$. Then $\alpha < \nu_s$ and $\bar{s} = s_\tau$ for the unique τ such that $J_s \models \tau = \alpha^+$. In addition, $f(\xi_\tau) \leqq \xi_\nu$. Moreover we have that $\delta_\eta \leqq \delta_\nu$ for η, ν as in ($\bar{B}$2) with $\alpha = \kappa$.

($\bar{B}$4) Let $\nu, f, \alpha, \bar{s}$ be as in ($\bar{B}$3). Assume that $\tau \in S_\kappa \cap \nu$ and $\tau \in \mathrm{rng}(f)$. Then $s_\tau \in \mathrm{rng}(f)$.

Lemma 2.4. *Let $\kappa > \omega$ be a cardinal and assume that ($\bar{B}$)$_\kappa$ holds. Then $\square_\kappa^{++}$ holds.*

Proof. Let ($\bar{B}$)$_\kappa$ be given by S, S_α, s_ν, ξ_ν, δ_ν. We have to define the various components of $\square_\kappa^{++}$. For $\alpha \leqq \kappa$ we take the same S_α, which have the right properties by ($\bar{B}$1). We now define C_ν. So let $\nu \in S_\alpha$, $s = s_\nu$. First we set:

$$\bar{C}_\nu = C_s - (\xi_\nu + 1).$$

For $\lambda \in \bar{C}_\nu$ set

$$X_{\nu,\lambda} = \mathrm{rng}\, f_{(\alpha, s \mid \lambda)} = h_{s \mid \lambda}(\alpha).$$

Then ($\bar{B}$2) yields that

$$\lambda \in \bar{C}_\nu \to X_{\nu,\lambda} \cap \nu \text{ is transitive.}$$

For $\lambda \in \bar{C}_\nu$ we set

$$\lambda^{(\nu)} = X_{\nu,\lambda} \cap \nu.$$

Using this notation we define

$$C_\nu = \{\lambda^{(\nu)} \mid \lambda \in \bar{C}_\nu\}.$$

Now let $\lambda \in C_\nu$, $\rho = \lambda^{(\nu)}$, $f = f_{(\alpha, s \mid \lambda)}$. Since $f(\xi_\rho) = \xi_\nu$ we have $f'' C_\rho = C_\nu \cap \lambda$. So (E1) follows as in the proof of Lemma 2.2. Now let $\nu \in S_\kappa$, $s = s_\nu$, $f_\alpha = f_{(\alpha,s)}$. We set

$$\bar{A}_\nu = \{\alpha < \kappa \mid \alpha = \beta(f_\alpha), \alpha \text{ no cardinal and}$$
$$(f_\alpha \text{ is cofinal or } \operatorname{otp}(C_s \cap \operatorname{rng} f_\alpha) \geqq \alpha)\}.$$

We shall set $A_\nu = \bar{A}_\nu - \bar{\delta}_\nu$ for some $\bar{\delta}_\nu < \kappa$ such that $\delta_\nu \leqq \bar{\delta}_\nu$. The exact definition of $\bar{\delta}_\nu$ will be given later. But we can already show that (E0)(d) holds. It suffices to show that if α is sufficiently large and $f_{(\alpha,s)}$ is not cofinal, then there is some $\beta < \alpha^+$ such that $\lambda(f_{(\beta,s)}) > \lambda(f_{(\alpha,s)})$. Let $H = H^{J_s}_{\kappa}$. So $H \in \operatorname{rng} f_{(\kappa,s)}$ by ($\bar{\text{B}}$0). Hence assume that $H \in \operatorname{rng} f_{(\alpha,s)}$ and f_α is not cofinal. Then

$$X = \{\langle i, \vec{\eta}\rangle \mid i \in \omega, \vec{\eta} < \alpha, \langle i, \vec{\eta}\rangle \in \operatorname{dom} h_s\} \in H.$$

So by ($\bar{\text{B}}$0) there is some $\beta < \alpha^+$ such that $X \in \operatorname{rng} f_{(\beta,s)}$. But then $\lambda = \lambda(f_{(\alpha,s)}) \in \operatorname{rng} f_{(\beta,s)}$ since λ is the smallest ordinal in J_s such that $X \subseteq \operatorname{dom} h_{s|\lambda}$.

Now we define G_α. So let $\nu \in S_\kappa$, $\alpha \in A_\nu$. Let $s = s_\nu$ and $f = f_{(\alpha,s)}$, $f: \bar{s} \Rightarrow s$. Then set

$$G_\alpha(\nu) = \text{the unique } \tau \text{ such that } J_{\bar{s}} \models \tau = \alpha^+.$$

Then $G_\alpha(\nu) \in S_\alpha$ by ($\bar{\text{B}}$3).

We now show (E2)(a). So let $\nu, \tau \in S_\kappa$, $\nu < \tau$, $s = s_\tau$. Since $f_{(\kappa,s)} = \mathrm{id}_s$ there is some $\eta < \kappa$ such that $\nu \in \operatorname{rng} f_{(\alpha,s)}$ for all $\alpha \in \Lambda_\tau - \eta$. Now let $\alpha \in A_\tau - \eta$, $f = f_{(\alpha,s)}$, $f: \bar{s} \Rightarrow s$. Then $s_\nu \in \operatorname{rng} f$ by ($\bar{\text{B}}$4). So let $f(\bar{s}) = s_\nu$. Set $s' = s_\nu$ and $\bar{f} = f \upharpoonright J_{\bar{s}}$. So $\bar{f}: \bar{s} \Rightarrow s'$. Since $\bar{s} \in J_s$ we have $\operatorname{rng} f_{(\alpha,s)} \in J_s$, $J_s \models |\operatorname{rng} f_{(\alpha,s)}| = \alpha$. But $\bar{f} \upharpoonright \alpha$ implies that $f_{(\alpha,s')} = \bar{f} \circ f_{(\alpha,\bar{s})}$. It follows easily that $G_\alpha(\nu) < G_\alpha(\tau)$.

We now turn to (E2)(b). But first we give the definition of $\bar{\delta}_\nu$. So let $\nu \in S_\kappa$, $s = s_\nu$. We set $\bar{\delta}_\nu = \max\{\delta_\nu, \delta'_\nu\}$ where δ'_ν is determined as follows. If $\operatorname{otp} C_s < \kappa$ set $\delta'_\nu = \operatorname{otp} C_s + 1$. Now let $\operatorname{otp} C_s = \kappa$. Then $D = C_s - C_\nu$ is some initial segment of C_s. Set $\delta'_\nu = \operatorname{otp} D + 1$. This definition of $\bar{\delta}_\nu$ guarantees that for (E2)(b) we only have to show

(*) Let $\alpha \in A_\nu$, $\eta = G_\alpha(\nu)$, $s = s_\nu$, $f = f_{(\alpha,s)}$. Let $\lambda \in C_\nu$ and assume that $f(\bar{\lambda}) = \lambda$. Then $G_\alpha(\lambda^{(\nu)}) = \bar{\lambda}^{(\eta)}$.

Note that $\bar{\lambda} \in C_\eta$ by ($\bar{\text{B}}$3) (since $f(\xi_\eta) \leqq \xi_\nu$), hence $\bar{\lambda}^{(\eta)}$ is defined.

First we introduce another notation. If $h: s' \Rightarrow s''$ we set $d(h) = s'$. Now we prove (*). Set $g = f_{(\kappa,s|\lambda)}$, $\bar{s} = s_\eta$, $\bar{g} = f_{(\alpha,\bar{s}|\bar{\lambda})}$, $\bar{\rho} = \bar{\lambda}^{(\eta)}$. By definition and ($\bar{\text{B}}$2) we know that

$$J_{d(g)} \models \rho = \kappa^+, \quad J_{d(\bar{g})} \models \bar{\rho} = \alpha^+, \quad d(g) = s_\rho, \quad d(\bar{g}) = s_{\bar{\rho}}.$$

Now set $\tilde{f} = f \upharpoonright J_{\bar{s}|\bar{\lambda}}$. So $\tilde{f}: \bar{s} \mid \bar{\lambda} \Rightarrow s \mid \lambda$ since $f(\bar{s} \mid \bar{\lambda}) = s \mid \lambda$. Since $\tilde{f} \upharpoonright \alpha = \mathrm{id} \upharpoonright \alpha$ we know that $\tilde{f} \circ \bar{g} = f_{(\alpha,s|\lambda)}$ and $g^{-1} \circ \tilde{f} \circ \bar{g} = f_{(\alpha,d(g))}$. But the last equation already gives that $G_\alpha(\rho)=\bar\rho$ since we get $d(f_{(\alpha,d(g))})=d(g)$ and we know that $d(g)=s_\rho$ and $J_{d(g)}\models\bar\rho=\alpha^+$. Note that $\alpha\in A_\rho$ by our definition of $\bar\delta_\rho$ and the last sentence of ($\bar{\text{B}}$3). qed

The following absoluteness property of $\square_\kappa$ has many important applications:

(*) Let W be an inner model and $\kappa>\omega$ some cardinal such that $\kappa^+=(\kappa^+)^W$. Then if $\square_\kappa$ holds in W, it holds in V, too.

This property is also true for $\square_\kappa^+$, but not for $\square_\kappa^{++}$, since successor cardinals below κ may not be the same in W and V. So we introduce another variation.

Definition. Let $\kappa>\omega$ be a cardinal.

(a) Let $A\subseteq\{\alpha\leqq\kappa\mid\lim(\alpha)\}$.
Then $\square_\kappa^{++}(A)$ denotes the variation of $\square_\kappa^{++}$ which results if we replace (E0)(d) by the following property:

Let $\nu\in S_\kappa$. Then there is some $\rho<\kappa$ such that $A_\nu\cap\eta$ is club in η for all $\eta\in A-\rho$.

(b) Let κ be singular. Then $\square_\kappa^+$ denotes the following principle:
There is $A\subseteq\kappa$ such that $\square_\kappa^{++}(A)$ holds and A has the following properties:
Let $\langle\kappa_\delta\mid\delta<\rho\rangle$ be the monotone enumeration of A. Then $\rho<\kappa$ and
(i) $\kappa>\kappa_\delta>\sup\{\kappa_\xi\mid\xi<\delta\}$ for all $\delta<\rho$,
(ii) $\langle\mathrm{cf}(\kappa_\delta)\mid\delta<\rho\rangle$ is weakly monotone,
(iii) $\kappa=\sup\{\kappa_\delta\mid\delta<\rho\}$.

Note that for singular κ, $\square_\kappa^+$ follows from $\square_\kappa^{++}$.

Clearly, the requirements for A in (b) are motivated by our specific applications. These requirements are just those appearing in the assumption of Lemma 1.3. Some of them are actually redundant in the definition of $\square_\kappa^+$.

Now $\square_\kappa^{++}(A)$ clearly satisfies (*). But this is still not the case for $\square_\kappa^+$ if κ is singular. But a weaker version of (*) holds which will be sufficient for us.

§3. $\square$-principles in K

In this section we show that $\square_\kappa^+$ holds in K for all cardinals $\kappa>\omega$. This will use the fine structure of the core model. Our main reference for this material will be [4]. But the basic properties will be used without explicit reference.

We only consider standard structures $N=J_\alpha^A$. We do not distinguish the structure $\langle J_\alpha^A,\in,A\cap J_\alpha^A\rangle$ and J_α^A. For $N=J_\alpha^A$ and $\lambda\leqq\alpha$ we set $N\mid\lambda=J_\lambda^A$.

Concerning general fine structure we use the same notation as in [4]. But we need a slight strengthening of Lemma 4.12 in [4]. For this we introduce a definition.

DEFINITION. Set M, N to be transitive, $\pi : M \to N$. Then $\pi : M \to_G N$ denotes that

(a) $\pi : M \to_{\Sigma_0} N$ and
(b) for cofinally many $\xi \in \mathrm{On} \cap M$ and all Σ_1-formulas ϕ we have $M \models A(\xi)$ iff $N \models \phi(\pi(\xi))$.

The following strengthening of Lemma 4.12 in [4] can be proved exactly as that lemma. So we omit the proof.

LEMMA 3.1. *Let N be acceptable, $\bar{M}$ transitive and $\pi : \bar{M} \to_G N^p$. Then there are unique M, $\bar{p}$ such that M is transitive, M is $\bar{p}$-sound and $\bar{M} = M^p$.*

In distinction to [4] we call every J_α a premouse, too. This seems to be a useful convention. The definitions for premice are extended in the obvious way. Especially, every J_α is a mouse. If $N = J_\alpha^U$ is a premouse and $N \neq J_\alpha$ we say that N is nontrivial. For premice $N = J_\alpha^U$ we also set

$$\mathrm{at}(N) = \begin{cases} \bigcup (U \cap N) & \text{if } N \text{ is nontrivial} \\ \omega\alpha & \text{otherwise} \end{cases}$$

and $\mathrm{lp}(N) = H^N_{\mathrm{at}(N)}$.

We also extend the natural well ordering of the core mice (see [4], Definition 15.7) to the class of all mice as follows:

Let M, N be mice. Then $M < N$ iff M, N have comparable mouse iterates M', N' with $M' \in N'$ or ($\mathrm{core}(M) = \mathrm{core}(N)$ and $\mathrm{at}(M) < \mathrm{at}(N)$). Formally, we define the core model by

$$K = \bigcup \{ \mathrm{lp}(M) \mid M \text{ a mouse} \}$$

and for $\nu \in \mathrm{On}$ we set

$$K_\nu = \bigcup \{ \mathrm{lp}(M) \mid M \text{ a mouse, } \mathrm{On} \cap M < \omega^\nu \}.$$

We shall need the following fact.

LEMMA 3.2. *Let N be a mouse at κ, $n = n(N)$, and $\bar{N} = N^n$. Let $\xi \in C_N$, $q \in [\xi]^{<\omega}$, $p = p_{\bar{N}} - \kappa$. Set $W = h_{\bar{N}}(q \cup p)$ and $\bar{W} = h_{\bar{N}}(q \cup p \cup \{\xi\})$.*

Then $W \cap \xi = \bar{W} \cap \xi$ *and* $\sup W = \sup \bar{W}$.

PROOF. Let M be the unique mouse at ξ such that N is a mouse iterate of M and set $\bar{M} = M^n$. So there is an iteration map $\pi : \bar{M} \to \bar{N}$. Note that $\pi(q) = q$ and $p \in \operatorname{rng} \pi$, say $\pi(\bar{p}) = p$. Now let $\sigma : \bar{Q} \xrightarrow{\sim} h_{\bar{M}}(q \cup \bar{p})$, $\bar{Q}$ transitive, and let $\sigma(\bar{\xi}) = \xi = \operatorname{at}(\bar{M})$. So $\sigma : \bar{M} \to_{\Sigma_1} \bar{M}$ and $W = \operatorname{rng}(\pi \circ \sigma)$. Let $\bar{\pi} : \bar{Q} \to_{\Sigma_1} Q$ be the 1-iteration. Then there is some $\bar{\sigma} : Q \to_{\Sigma_1} \bar{N}$ such that $\pi \circ \sigma = \bar{\sigma} \circ \bar{\pi}$ and $\bar{\sigma}(\xi) = \xi$. Then $\bar{W} = \operatorname{rng} \bar{\sigma}$ since $Q = h_Q(\operatorname{rng}(\bar{\pi}) \cup \{\xi\})$. Hence the claim follows since $\bar{\pi} \upharpoonright \bar{\xi} = \operatorname{id} \upharpoonright \xi$ and $\bar{\pi}$ is cofinal. qed

We also need an additional lemma about the fine structure of mice.

LEMMA 3.3. *Let N be a mouse, $\bar{M}$ transitive and $\pi : \bar{M} \to_G N^m$. Then there is a unique m-sound iterable premouse M such that $\bar{M} = M^m$. Moreover, if $m > n(N)$ then M is a core mouse and $n(M) = n(N)$.*

PROOF. We do induction on m. For $m \leqq n(N)$ this is clear using soundness above the critical point (see [4], Lemma 9.7), Lemma 3.1 and standard arguments. Note that for $m = 0$ the fact that π is a G-embedding is sufficient to get that $\bar{M}$ is a premouse. Now let $m = n(N) + 1$. This is the critical case. Set $n = n(N)$. We may assume that N is a core mouse since $N^{n+1} = \operatorname{core}(N)^{n+1}$. Set $Q = N^n$, $p = p_N^{n+1}$. By Lemma 3.1 there is some transitive $\bar{Q}$ and some $\bar{p}$, $\bar{Q}$, $\bar{p}$-sound, such that $\bar{M} = \bar{Q}^{\bar{p}}$. Let $\tilde{\pi} : \bar{Q} \to_{\Sigma_1} Q$, $\tilde{\pi}(\bar{p}) = p$, be the canonical extension of π. By induction hypothesis there is some iterable premouse M such that $\bar{Q} = M^n$. The embedding $\tilde{\pi}$ shows that M is a mouse and $n(M) = n$. Since $\bar{Q}$ is $\bar{p}$-sound it suffices to show that $\bar{p}$ is the standard parameter of $\bar{Q}$. The usual argument using $\tilde{\pi}$ shows that $\bar{p}$ is the $<_*$-least parameter q such that $h_{\bar{Q}}(\rho_{\bar{Q}} \cup q) = \bar{Q}$. But then it is known that $\bar{p} = p_M^{n+1} \cup C_M$. So we have to show that $C_M = \varnothing$. Assume that $C_M \neq \varnothing$. We shall derive a contradiction. Since π is Σ_1-preserving for certain parameters, $\tilde{\pi}$ is Π_2-preserving for certain parameters. Especially we get

(*) $\bar{Q} \models \phi(\bar{p}) \to Q \models \phi(p)$ (ϕ Π_2-formula).

We apply this to the statements in Lemma 3.2. So for $\xi \in \tilde{\sigma}'' C_M$, $r = p - \operatorname{at}(N)$ we get

$$\forall q \in [\xi]^{<\omega} \qquad h_Q(q \cup r \cup \{\xi\}) \cap \xi \subseteq h_Q(q \cup r).$$

This eventually yields that $\xi = \min \tilde{\sigma}'' C_M$ and $q = p - \{\xi\}$ satisfy $h_Q(\rho_Q \cup q) \subseteq \xi$. But (*) also implies that ξ is a cardinal in Q. So r defines a new Σ_1-subset of $\omega\rho_Q$ in Q. This is a contradiction since $r <_* p$ and p is the standard parameter of Q. This ends the proof for $m = n(N) + 1$. But the rest of the induction is clear by the m-soundness of core mice. qed

Later we shall also use the fact that if in the lemma above $m = n(N)$, then M will be a mouse if we only know that M is critical.

After these preliminaries we now come to the main part of this section. Set

$$\tilde{S} = \{\nu \mid \nu \text{ primitive recursively closed, } \nu \text{ no cardinal in } K\}.$$

For $\nu \in \tilde{S}$ set

$N(\nu) =$ the $<$-least mouse N such that $\mathrm{at}(N) \geqq \nu$ and ν is no Σ_ω-cardinal in N.

$N(\nu)$ is called the minimal collapsing mouse for ν.

LEMMA 3.4. *Let $\nu \in \tilde{S}$ and $N = N(\nu)$. Then N is uniquely characterized by the following three properties*:

(i) *N is a mouse*, $\mathrm{at}(N) \geqq \nu$, *ν is no Σ_ω-cardinal in N.*

(ii) *ν is a cardinal in N.*

(iii) $C_N \subseteq \nu$.

PROOF. First we show that N satisfies (i)–(iii). For (i) this is clear by definition. Let $N = J^U_\alpha$, $\mathrm{at}(N) = \kappa$. Now assume that ν is no cardinal in N. Then $\nu < \kappa$. Let $f \in N$ show that ν is no cardinal in N. Let $f \in J^U_{\beta+1} - J^U_\beta$. But then it is known that $f \in \Sigma_\omega(J^U_\beta)$ (see [4], Corollary 11.10) and J^u_β is a mouse. This contradicts the minimality of N. Now assume that $C_N \not\subseteq \nu$. Let $\bar{\kappa} \in C_N - \nu$. So there is a mouse M at $\bar{\kappa}$ such that N is a mouse iterate of M. But $\mathfrak{p}(\bar{\kappa}) \cap \Sigma_\omega(N) \subseteq M$. So ν is no Σ_ω-cardinal in M. This contradicts minimality again.

The usual comparability argument shows that N is the only mouse satisfying (i)–(iii). qed

We now define $S \subseteq \tilde{S}$ as follows.

Let $\nu \in \tilde{S}$. Then $\nu \in S$ iff the following three properties hold:

(i) $N(\nu) \models \nu = \alpha^+$ for some $\alpha > \omega$.

(ii) $N(\nu) \models H_\nu = K_\nu$.

(iii) $H^{N(\nu)}_\nu \models \forall x\ \exists y (x \in y$ and y is admissible).

For $\nu \in S$ we set

$$\alpha_\nu = \text{the unique } \alpha \text{ such that } N(\nu) \models \nu = \alpha^+,$$
$$H(\nu) = H^{N(\nu)}_\nu.$$

Further let

$$S_\alpha = \{\nu \in S \mid \alpha_\nu = \alpha\}.$$

By simple absoluteness arguments we get

REMARK 3.5. Let $\nu \in S_\alpha$. Then $\langle N(\tau) \mid \tau \in S_\alpha \cap \nu \rangle$ is (uniformly) Σ_1-definable in $H(\nu)$ with the parameter α. We now show

LEMMA 3.6. *Assume $V = K$. Then*
(a) $S_\alpha \subseteq \alpha^+$, *$S_\alpha$ is closed in α^+*,
(b) *α no cardinal* $\rightarrow \sup S_\alpha < \alpha^+$,
(c) *$\kappa > \omega$ a cardinal* $\rightarrow \sup S_\kappa = \kappa^+$.

PROOF. (a) Obviously, $S_\alpha \subseteq \alpha^+$. Now let $\nu < \alpha^+$ be a limit point of S_α. Set $H = \bigcup\{H(\tau) \mid \tau \in S_\alpha \cap \nu\}$. Clearly, $\nu \in \bar{S}$. So let $N = N(\nu)$. It is easy to see that we only have to show that $H = H^N_\nu$. Actually, $H \subseteq H^N_\nu$ is obvious and it suffices to show that $H^N_\nu \cap \mathfrak{p}(\nu) \subseteq H$. So let $a \in N$ be a bounded subset of ν. Let $N = J^U_\beta$. So $a \in J^U_{\gamma+1} - J^U_\gamma$ for some $\gamma < \beta$. Hence $M = J^U_\gamma$ is a mouse and $\rho^n_M < \nu$ for some n. But ν is Σ_ω-regular in M. Hence $\nu \in C_M$ and ν is a limit cardinal in M. So there is some $\tau \in S_\alpha \cap \nu$ such that $M < N(\tau)$. This implies $a \in H$.

(b) Choose a mouse M such that $|M| < \alpha^+$, $\mathrm{at}(M) > \alpha$ and α is no cardinal in M. Obviously, $\sup S_\alpha < \mathrm{at}(M)$.

(c) If $\kappa > \omega$ is a cardinal then $\{\nu < \kappa^+ \mid K_\nu \prec K_{\kappa^+}\} \subseteq S_\kappa$. qed

For $\nu \in \bar{S}$, $N = N(\nu)$, set $n = n(\nu) =$ the least n such that ν is no Σ_{n+1}-cardinal in N; $A(\nu) = A^n_N$, $\rho(\nu) = \rho^n_N$, $\bar{N}(\nu) = N^n$.

LEMMA 3.7. *Let $\nu \in S$, $\alpha = \alpha_\nu$, $\bar{N} = \bar{N}(\nu)$. Then there is some $p \in \bar{N} \cap [\mathrm{On}]^{<\omega}$ such that $\bar{N} = h_{\bar{N}}(\alpha \cup p)$.*

PROOF. Let $m = n(N)$. Obviously, $n(\nu) = n \geqq m$. First assume that $n = m$. Since N is m-sound we know that ν is a cardinal in $\bar{N}$ but no Σ_1-cardinal in $\bar{N}$. Hence $\rho_{\bar{N}} \leqq \alpha$ since $\bar{N} \models \nu = \alpha^+$. We also know that $C_N \subseteq \nu$. So $N \models \nu = \alpha^+$ implies that $C_N \subseteq \alpha + 1$. So the claim follows from $\bar{N} = h_{\bar{N}}(\rho_{\bar{N}} \cup p_{\bar{N}} \cup C_N)$ (see [4], Lemma 10.19). If $n > m$ then N is a core mouse. So N is $n+1$-sound and the claim follows easily. qed

Now let $\nu \in S$, $\alpha = \alpha_\nu$, $\bar{N} = \bar{N}(\nu)$, $N = N(\nu)$, $\rho = \rho(\nu)$, $m = n(N)$, $n = n(\nu)$. By Lemma 3.7 we can define

$$p = p(\nu) = \text{the } <_*\text{-least } p \in \bar{N} \text{ such that } \bar{N} = h_{\bar{N}}(\alpha \cup p).$$

We set $\hat{\nu} = \nu$, if $\nu < \rho$, and $\hat{\nu} = 0$ otherwise. We need one additional parameter $r(\nu)$. If $n > m$, we set $r(\nu) = 1$. So let $n = m$ and $\kappa = \mathrm{at}(N) = \mathrm{at}(\bar{N})$.

If $p \cap [\nu, \kappa) = \varnothing$ we set $r(\nu) = 1$. Otherwise, let $\xi = \max(p \cap [\nu, \kappa))$. If ξ is singular in N we set $r(\nu) = 1$, too. So let ξ be regular in N. Note that in our case N is a core mouse and $p = p_{\bar{N}} - \alpha$. For $\mu < \alpha$ and $\max(p) < \lambda \leqq \rho$ set $\bar{W}^{\lambda}_{\mu} = h_{\bar{N}|\lambda}(\{\mu\} \cup p)$, $W^{\lambda}_{\mu} = h_{\bar{N}|\lambda}(\{\mu\} \cup (p - \{\xi\})$. We also set $\bar{W}_{\mu} = W^{\rho}_{\mu}$ and $W_{\mu} = W^{\rho}_{\mu}$.

Now set $I = \{\mu < \alpha \mid \forall \bar{\mu} < \mu \sup \bar{W}_{\bar{\mu}} < \sup \bar{W}_{\mu}\}$ and

$$\bar{I} = \{\mu \in I \mid \sup W_{\mu} < \sup \bar{W}_{\mu} \text{ or } \exists \lambda \in W_{\mu}\ \exists \gamma < \alpha \bar{W}^{\lambda}_{\gamma} \cap \xi \not\subseteq W^{\lambda}_{\gamma}\}.$$

We now show that $\bar{I} \neq \varnothing$.

Proof. Assume that $\bar{I} = \varnothing$. Set $p' = p - \{\xi\}$. We first show that $h_{\bar{N}}(\alpha \cup p') \supseteq \xi$. For notational reasons assume that ρ is a limit ordinal. So let $\delta < \xi$. Since $\delta \in h_{\bar{N}}(\alpha \cup p)$ there are $\gamma < \alpha$ and $\eta < \rho$ such that $\delta \in \bar{W}^{\eta}_{\gamma}$. Since $\bar{I} = \varnothing$ there are $\mu \in I$ and $\lambda \in W_{\mu}$ such that $\lambda \geqq \eta$. Using $\bar{I} = \varnothing$ again we get $\delta \in \bar{W}^{\eta}_{\gamma} \cap \xi \subseteq W^{\lambda}_{\gamma}$. This shows $h_{\bar{N}}(\alpha \cup p') \supseteq \xi$. Since ξ is regular in $\bar{N}$ there is some B such that $B \cap \alpha \notin \bar{N}$ and B is Σ_1-definable in $\bar{N}$ with parameters from $\alpha \cup p'$. Now let $\sigma : \bar{M} \xrightarrow{\sim} h_{\bar{N}}(\alpha \cup p')$, $\bar{N}$ transitive, and $\sigma(\bar{p}) = p'$. Then $\bar{M} = h_{\bar{M}}(\alpha \cup \bar{p})$ and $\bar{M} = M^m$ for some mouse M. Clearly, $M \leqq N$. But B is Σ_1-definable in M. Hence $M = N$ since N is a core mouse. So $\bar{M} = \bar{N}$ and $\bar{N} = h_{\bar{N}}(\alpha \cup \bar{p})$. This contradicts the minimality of p since $\bar{p} <_{*} p$. qed

So we can define $r(\nu) = \min \bar{I}$. Note that $r(\nu) < \alpha$.

Finally, we set

$$q(\nu) = p(\nu) \cup \{\alpha, \hat{\nu}, r(\nu)\},$$

$$s_{\nu} = \langle \rho(\nu), a(\nu) \rangle, \qquad \text{where } a(\nu) = (A(\nu) \times \{0\}) \cup (q(\nu) \times \{1\}).$$

So $s = s_{\nu} \in \mathcal{S}$, $f_{(\alpha,s)} = \mathrm{id}_s$ and $\bar{N} = J_s$ (as sets).

Theorem 3.8. *Assume $V = K$ and let $\kappa > \omega$ be regular. Then $\langle s_{\nu} \mid \nu \in S \cap \kappa^{+} \rangle$ satisfies* $(\mathrm{B})_{\kappa}$.

Proof. First note that the "S_{α}-notation" above is the same as in the definition of $(\mathrm{B})_{\kappa}$. So (B0), (B1) have already been shown.

(B2) Let $\nu \in S_{\alpha}$, $s = s_{\nu}$, $\lambda \in C_s$, $f = f_{(\alpha, s|\lambda)}$, $f : \bar{s} \Rightarrow s \mid \lambda$ and $\eta = \sup\{\rho \leqq \nu_s \mid f(\rho) \leqq \nu\}$. We have to show that $\eta \in S_{\alpha}$ and $\bar{s} = s_{\eta}$. Set $\bar{N} = \bar{N}(\nu)$, $n = n(\nu)$, $p = p(\nu)$, $q = q(\nu)$, $r = r(\nu)$. Recalling the definition of C_s it is easy to see that

(1) $q \in \operatorname{rng} f$, $f(\alpha) = \alpha$, $f(r) = r$, $f : J_{\bar{s}} \to_G J_s$ and $f(\eta) = \nu$, if $\nu < \rho$.

Let $a_{\bar{s}} = (\bar{A} \times \{0\}) \cup (\bar{q} \times \{1\})$, $f(\bar{p}) = p$, $\omega\bar{\rho} = \nu_{\bar{s}}$, $\bar{M} = J^{\bar{A}}_{\bar{\rho}}$.

(2) $h_{\bar{M}}(\alpha \cup \bar{p}) = \bar{M}$.

PROOF. By definition of f we know that $h_{\bar{M}}(\alpha \cup \bar{q}) = \bar{M}$. But $f: J_{\bar{s}} \rightarrow_G J_s$ implies that $\bar{q} \in h_{\bar{M}}(\alpha \cup \bar{p})$, since $q \in h_{\bar{N}}(\alpha \cup p)$. qed (2)

Clearly, $f: \bar{M} \rightarrow_G \bar{N}$. Hence by Lemma 3.3 and the following remark there is an m-sound mouse M such that $\bar{M} = M^n$. We also have

(3) $n(M) = n(N)$, $\mathrm{at}(M) \geqq \eta$, η is a Σ_n-cardinal but no Σ_{n+1}-cardinal in M. We now show

(4) $\bar{p} =$ the $<^*$-least p' such that $h_{\bar{M}}(\alpha \cup p') = \bar{M}$.

PROOF. Otherwise there is some $p' <_* \bar{p}$ such that $\bar{p} \in h_{\bar{M}}(\alpha \cup p')$. But then $p \in h_{\bar{N}}(\alpha \cup f(p'))$ and $f(p') <_* p$. Contradiction. qed (4)

(5) $C_M \subseteq \eta$.

PROOF. If $n > n(N)$ then M is a core mouse, hence $C_M = \varnothing$. So let $n = n(N)$ and set $\tilde{\kappa} = \mathrm{at}(N)$, $\bar{\kappa} = \mathrm{at}(M)$, so $f(\bar{\kappa}) = \tilde{\kappa}$. First assume $p \cap [\nu, \tilde{\kappa}) = \varnothing$. Then $\bar{p} \cap [\eta, \bar{\kappa}) = \varnothing$. But then by known properties of mice (4) implies that $C_M \subseteq \eta$. So let $p \cap [\nu, \tilde{\kappa}) \neq \varnothing$. For this case we introduced the additional parameter $r(\nu)$. Let $\xi = \max(p \cap [\nu, \tilde{\kappa}))$, $f(\bar{\xi}) = \xi$. We shall show that $C_M = \varnothing$. If this were not the case then again (4) implies that $\bar{\xi} \in C_M$, $p_{\bar{M}} - \bar{\kappa} = \bar{p} - \bar{\kappa}$ and $\bar{\xi}$ is regular in $\bar{M}$. But we know that

$$\bar{M} \models \phi(\bar{p}, r) \qquad \text{iff } \bar{N} \models \phi(p, r) \quad (\phi\ \Sigma_1\text{-formula}).$$

So by definition of r

$$\sup h_{\bar{M}}(\{r\} \cup (\bar{p} - \{\bar{\xi}\})) < \sup h_{\bar{M}}(\{r\} \cup \bar{p})$$

or $\exists \lambda \in h_{\bar{M}}(\{r\} \cup (\bar{p} - \{\bar{\xi}\}))\ \exists \gamma < \bar{\xi}$

$$h_{\bar{M}|\lambda}(\{\gamma\} \cup \bar{p}) \cap \bar{\xi} \not\subseteq h_{\bar{M}|\lambda}(\{\gamma\} \cup (\bar{p} - \{\bar{\xi}\})).$$

But by Lemma 3.2 this shows that $\bar{\xi} \notin C_M$.

Putting all this together we have already shown that $\eta \in \bar{S}$, $M = N(\eta)$ and $\bar{M} = \bar{N}(\eta)$. Using the fact that $f: J_{\bar{s}} \rightarrow_G J_s$ it is easy to check that

(6) $\eta \in S_\alpha$.

Hence by (4) we also know that $\bar{p} = p(\eta)$.

(7) $r = r(\eta)$.

PROOF. We only have to treat the case where $r(\nu)$ has been defined in a nontrivial way. But since $\lambda \in C_s$ we have $h_{\bar{N}}(\{r\} \cup p) \subseteq \bar{N} \mid \lambda$. Since $r \in I$ we can replace r by any $\mu \leqq r$. Hence we have for all $\mu \leqq r$

$$\bar{M} \models \phi(\mu, \bar{p}) \qquad \text{iff } \bar{N} \models \phi(\mu, p) \quad (\phi\ \Sigma_1\text{-formula}).$$

This implies the claim. qed (7)

Putting all this together we have $\bar{s} = s_\eta$. qed (B2)

The proof of (B3) is almost a literal translation of the arguments above. Actually, it is a little bit simpler since we have a full Σ_1-embedding. So we leave this to the reader.

(B4) Let f be given, $f(\bar{\tau}) = \tau$, $\tau \in S_\kappa \cap \nu$. So by Remark 3.5 $N(\tau) \in \operatorname{rng} f$. Let $f(M) = N(\tau)$. So $f \restriction M : N \to_{\Sigma_\omega} N(\tau)$ and $f(\alpha) = \kappa$. It follows easily that $\bar{\tau} \in S_\alpha$, $M = N(\bar{\tau})$ and $f(s_{\bar{\tau}}) = s_\tau$. qed

Now we turn to the principle $(\bar{\mathrm{B}})_\kappa$. In general, this principle is no longer true in K for all cardinals $\kappa < \omega$. For example, the results of the next section show that $(\bar{\mathrm{B}})_\kappa$ fails in K if κ is measurable in an inner model. In L, however, $(\bar{\mathrm{B}})_\kappa$ is true for all cardinals $\kappa > \omega$. For our application we only have to consider singular cardinals κ. So let κ be a singular cardinal in K. Then we set

$$S^*_\kappa = \{\nu \in S_\kappa \mid N(\nu) \models \kappa \text{ is singular}\}.$$

Obviously, S^*_κ is a nonempty final segment of S_κ. A standard argument shows that $K_\kappa \in N(\nu)$ for $\nu \in S^*_\kappa$. For $\alpha < \kappa$ set

$$S^*_\alpha = \begin{cases} S_\alpha & \text{if } \alpha \text{ is no cardinal in } K, \\ \varnothing & \text{otherwise.} \end{cases}$$

Then set $S^* = \bigcup\{S^*_\alpha \mid \alpha \leqq \kappa\}$. For $\nu \in S^*$ we now set

$$p^*(\nu) = \{r(\nu)\} \cup p(\nu)$$

and

$$s^*_\nu = \langle \rho(\nu), a^*(\nu) \rangle, \qquad \text{where } a^*(\nu) = (A(\nu) \times \{0\}) \cup (p^*(\nu) \times \{1\}).$$

So we only change the parameter.

We need two technical lemmas.

LEMMA 3.9. *Let* $\nu \in S^*$, $N = N(\nu) = J_\beta^U$, $p = p(\nu)$, $\alpha = \alpha_\nu$, $H = H(\nu)$. *Assume that* $n(\nu) = 0$, $\lim(\beta)$, $\nu = \mathrm{at}(N)$ *and* $H \notin N$. *Then there is some* $\delta \in p$ *such that* $\alpha = |\delta|^N$.

PROOF. The easy argument used in the proof of Lemma 6.5 in [5] shows that $h_N(\nu) = N$, since $H \notin N$. So we have $p \subseteq \nu$. Now assume that the claim is false. Then $p = \varnothing$, since $p \subseteq \nu - \alpha$. We derive a contradiction by showing that $h_N(\alpha) \cap \nu \subseteq \alpha$. So let $\delta \in h_N(\alpha) \cap \nu$. Hence there is a Σ_0-formula ϕ and a $\gamma < \alpha$ such that

(∗) $\delta = x$ iff $N \models \exists z\, \phi(z, \gamma, x)$.

Choose $\eta < \beta$ minimal such that $J_\eta^U \models \exists z, x\, \phi(z, \gamma, x)$. Set $M = J_\eta^U$. A standard argument shows that $h_M(\nu) = M$. So M is a mouse and $n(M) = 0$. Let $\bar{M}$ be transitive, $\bar{M} \xrightarrow{\sim} h_M(\gamma + 1)$. Then $\bar{M}$ is a mouse and $\bar{M} \leqq M$. But the minimal choice of η guarantees that $\mathrm{core}(M) = \mathrm{core}(\bar{M})$. So M is an iterate of $\bar{M}$. Let $\pi : \bar{M} \to M$ be the iteration map. Then (∗) implies that $\delta \in \mathrm{rng}(\pi) \cap \nu \subseteq \bar{M}$. But $\mathrm{On} \cap \bar{M} < \alpha$ by definition of $\bar{M}$. Hence $\delta < \alpha$. qed

LEMMA 3.10. *Let* $\nu \in S^*$, $\nu < \mathrm{at}(N(\nu))$ *and* $\mathrm{cf}(\nu) > \omega$. *Then* $H(\nu) \in N(\nu)$.

PROOF. Set $H = H(\nu)$ and let $Q = \bigcup\{M_\nu \mid M \in H, M \text{ mouse}\}$ where M_ν denotes the ν-th iterate of M. Let $F = \{E \subseteq \nu \mid E \supseteq G$ for some club $G \subseteq \nu\}$. Then $Q = J_\delta^F$ for some δ, Q is a countably complete premouse and $H \subseteq Q$. Choose $\tilde{\delta} \geqq \delta$ minimal such that $\tilde{Q} = J_{\tilde{\delta}}^F$ is critical. Then $\tilde{Q}$ is a mouse since $\tilde{Q}$ is countably complete. Hence $\tilde{Q} \leqq N(\nu)$. But $N(\nu) \neq \tilde{Q}$ since at $(N(\nu)) > \nu$. Hence $H(\nu) \in N(\nu)$. qed

THEOREM 3.11. *Assume* $V = K$ *and let* κ *be a singular cardinal. Then* $\langle s_\nu^* \mid \nu \in S^* \rangle$ *satisfies* $(\bar{\mathrm{B}})_\kappa$.

PROOF. ($\bar{\mathrm{B}}$0), ($\bar{\mathrm{B}}$1) are clear. From now on we use the "S_α^*-notation".

($\bar{\mathrm{B}}$2) First we define $\langle \xi_\nu \mid \nu \in S^* \rangle$. So let $\nu \in S^*$, $s^* = s_\nu^*$, $C = C_{s^*}$, $\rho = \rho(\nu)$, $N = N(\nu)$ etc. We distinguish two cases.

Case 1. $n = 0$ and (ρ is a successor ordinal or $\nu < \mathrm{at}(N)$ and $H \notin N$)). In this case $\mathrm{cf}(\nu) = \omega$ (especially, see Lemma 3.10). We set $\xi_\nu = \rho$. So ($\bar{\mathrm{B}}$2) is trivially satisfied.

Case 2. Case 1 is not satisfied. We set

$$\xi_\nu = \begin{cases} \min\{\lambda \in C \cup \{\rho\} \mid H \in h_{N|\lambda}(\alpha \cup p)\} & \text{if } n = 0 \text{ and } H \in N, \\ \min\{\lambda \in C \cup \{\rho\} \mid \alpha \in h_{\bar{N}|\lambda}(\alpha \cup p)\} & \text{otherwise.} \end{cases}$$

Note that in any case $\alpha \in h_{\bar{N}|\xi_\nu}(\alpha \cup p)$ because α is definable in H as the largest cardinal. Clearly, we have $\xi_\nu < \rho$ if $\sup C = \rho$. Hence ($\bar{\mathrm{B}}$2)(a) is satisfied. We now show ($\bar{\mathrm{B}}$2)(b). So let $\lambda \in C$, $\lambda > \xi_\nu$, $f = f_{(\alpha, s^* | \lambda)}$, $f: \bar{s} \Rightarrow s^* \mid \lambda$ and set $\eta = \sup\{\delta \leqq \nu_{\bar{s}} \mid f(\delta) \leqq \nu\}$. First we show that $\eta \in S_\alpha$ and $\bar{s} = s^*_\eta$. This follows almost exactly as in the proof of (B2) in Theorem 3.8. By definition of ξ_ν, we have $f(\alpha) = \alpha$. Especially, $\bar{s}$, f given in a canonical way $\bar{M}$, M such that $\bar{M} = \bar{N}(\eta)$, $M = N(\eta)$. To show that $\eta \in S_\alpha$ we distinguish several cases. If $n > 0$, then there is a canonical extension $\tilde{f} \supseteq f$ such that $\tilde{f}: M \rightarrow_{\Sigma_1} N(\nu)$. Then $\eta \in S_\alpha$ follows easily. So let $n = 0$. If $H \notin N$, hence $\nu = \mathrm{at}(N)$, then by Lemma 3.9 α is Σ_1 = definable in N with the parameter p because N has a Σ_1 Skolem function. But then $\eta \in S_\alpha$ follows exactly as in Theorem 3.8. Hence finally let $n = 0$ and $H \in N$. Then $H \in \mathrm{rng} f$ by definition of ξ_ν. So let $f(\bar{H}) = H$. It follows easily that $H = H^M_\eta$ and $\eta \in S_\alpha$. If $\alpha = \kappa$, then it is easy to see that $N \mid \lambda \vDash \kappa$ is singular. Hence we have $\eta \in S^*_\alpha$.

The only thing left to show is $f(\xi_\eta) = \xi_\nu$. This is obvious if $n > 0$ or $H \in N$. So let $n = 0$, $H \notin N$ and $\nu = \mathrm{at}(N)$. We have to show that $H(\eta) \notin N(\eta)$. But in our case α is Σ_1-definable in N with the parameter p. So $f: N(\eta) \rightarrow N$ is Σ_1 with respect to the parameter α. It follows that $\mathfrak{p}(\alpha) \cap N(\eta) \notin N(\eta)$ because $H \notin N$ and α is the largest cardinal in N. qed ($\bar{\mathrm{B}}$2)

($\bar{\mathrm{B}}$3) First we define $\langle \delta_\nu \mid \nu \in S^*_\kappa \rangle$. So let $\nu \in S^*_\kappa$. We set

$$\bar{\delta}_\nu = \min\{\delta < \kappa \mid K_\kappa \in h_{\bar{N}|\xi_\nu}(\delta \cup p)\} \quad \text{and} \quad \delta_\nu = \max\{\bar{\delta}_\nu, r(\nu) + 1\}.$$

So the last claim in ($\bar{\mathrm{B}}$3) is satisfied. Now let $\nu \in S^*_\kappa$, $s^* = s^*_\nu$, $\delta_\nu \leqq \alpha < \kappa$, α no cardinal, $f = f_{(\alpha, s^*)}$, $f: \bar{s} \Rightarrow s^*$, $\alpha = \beta(f)$. Since $\kappa \in \mathrm{rng} f$, we have $\alpha < \nu_{\bar{s}}$. So let $J_{\bar{s}} \vDash \tau = \alpha^+$. Let $\bar{M} = J^{\bar{A}}_{\bar{\rho}}$, where $a_{\bar{s}} = (\bar{A} \times \{0\}) \cup (\bar{q} \times \{1\})$, $\bar{\rho} = \nu_{\bar{s}}$ and let $f(\bar{p}) = p$. Note that $f(r) = r$.

Exactly as before we get

(1) $h_{\bar{M}}(\alpha \cup \bar{p}) = \bar{M}$.

(2) There is an m-sound mouse M such that $M^n = \bar{M}$, $n(M) = n(N)$, $\mathrm{at}(M) \geqq \tau$. Moreover τ is a Σ_n-cardinal in M but no Σ_{n+1}-cardinal in M.

(3) $\bar{p}$ = the $<_*$-least p' such that $h_{\bar{M}}(\alpha \cup p') = \bar{M}$.

But we have to use one new argument to show

(4) $C_M \subseteq \tau$.

PROOF. As before we may assume that $n = n(N)$. Moreover $C_M - \tau \subseteq \bar{p}$. The parameter r shows again that $\gamma \notin C_M$ if $f(\gamma) \geqq \nu$. But if $\gamma \in \bar{p}$ and $f(\gamma) < \nu$, then $|f(\gamma)|^N = \kappa$. So in this case $f(\gamma)$ is singular in N, hence γ is singular in M and $\gamma \notin C_M$. qed (4)

It is easy to see that

(5) $\tau \in S^*_\alpha$.

We now show

(6) $r = r(\tau)$.

PROOF. Since κ is singular in N it is easy to see that $r = 1$ iff $r(\tau) = 1$. So let $r \neq 1$, hence r has been defined in a nontrivial way. Since $f \upharpoonright r + 1 = \mathrm{id} \upharpoonright r + 1$ and f is Σ_1-elementary we immediately get that $r = r(\tau)$. qed (6)

Putting all this together we have shown $\bar{s} = s_\tau$.

Finally, we show $f(\xi_\tau) \leqq \xi_\nu$. It suffices to show:

(7) Let $\lambda \in C_s$ and $f(\lambda) \geqq \xi_\nu$. Then α, $H(\tau) \in h_{\bar{M}|\lambda}(\alpha \cup \bar{p})$.

PROOF. We know that $K_\kappa \in \mathrm{rng}\, f$. So let $f(\bar{K}) = K_\kappa$. Obviously, $\bar{K} = H^M_{\bar{\kappa}}$, where $f(\bar{\kappa}) = \kappa$. Actually, we know that $\bar{K} \in h_{\bar{M}|\lambda}(\alpha \cup p)$, since $f(\lambda) \geqq \xi_\nu$. Set $\gamma = |\alpha| < \alpha$. Clearly, $\alpha = (\gamma^+)^M$. Hence α, $H(\tau)$ are definable in $\bar{K}$ with the parameter γ. Hence $\bar{K} \in h_{\bar{M}|\lambda}(\alpha \cup p)$ implies the claim. qed ($\bar{\mathrm{B}}$3)

($\bar{\mathrm{B}}$4) is obvious. qed

Using the covering theorem for K we get

THEOREM 3.12. *Assume $\neg L^\mu$. Let κ be a singular cardinal. Then $\square^+_\kappa$ holds.*

PROOF. By the covering theorem we know that κ is singular in K and $\kappa^+ = (\kappa^+)^K$. Choose $\bar{A} \subseteq \mathrm{Card} \cap \kappa$ such that $\sup \bar{A} = \kappa$, $|\bar{A}| < \kappa$ and $\omega_2 \leqq \min \bar{A}$. Set $A = \{(\tau^+)^K \mid \tau \in \bar{A}\}$. By the covering theorem A satisfies the conditions (i)–(iii) in the definition of $\square^+_\kappa$. By Lemma 2.4 and Theorem 3.11 there is a $\square^{++}_\kappa$-sequence in K. This sequence is a $\square^{++}_\kappa(A)$-sequence in V. qed

We also need the following facts about "partial measures" in K.

LEMMA 3.13. *Let $\nu \in S_\kappa$. Set $H = H(\nu)$, $N = N(\nu)$.*

(a) *Assume that $U \subseteq \mathfrak{p}(\kappa) \cap H$ and $\langle H, U \rangle \models$ "U is normal", $\langle H, U \rangle$ is amenable, and U is countable complete. Then $\kappa \in C_N$.*

(b) *Assume that* $\kappa \in C_N$. *Then there is some* $U \subseteq \mathfrak{p}(\kappa) \cap H$ *such that* $\langle H, U \rangle$ *is amenable and* $\langle H, U \rangle \models$ "*U is normal*".

(c) *Let* $U_i \subseteq \mathfrak{p}(\kappa) \cap H$ $(i = 0, 1)$ *such that* $\langle H, U_i \rangle \models$ "U_i *is normal*", $\langle H, U_i \rangle$ *is amenable and* U_i *is countably complete. Then* $U_0 = U_1$.

PROOF. (a) Set $\bar{M} = \bigcup\{ J^U_{\gamma+1} \mid J^U_\gamma \in N\}$. So $\bar{M} = J^U_\delta$ for some δ. By the assumptions we get

(1) $\bar{M}$ is a premouse, $\bar{M} \subseteq H$, $\bar{M} \notin H$.

Since $\nu < (\kappa^+)^K$ we know that U is not normal in $L[U]$. Hence there is some $\beta \geqq \delta$ such that $\mathfrak{p}(\kappa) \cap \Sigma_\omega(J^U_\beta) \not\subseteq J_\beta$. Let β be the least such and set $M = J^U_\beta$. Such an M is a mouse since U is countably complete. A standard argument shows that

(2) For all mice $P \in H$ we have $P < M$.

Now let M_1 be the 1-mouse iteration of M. It suffices to show that $M_1 = N$. We use Lemma 3.4. Since $H \models V = K$ and since mice in the sense of H are really mice, we get by (2) that $\mathrm{at}(M_1) \geqq \nu$. Obviously, ν is no Σ_ω-cardinal in M_1 and $C_{M_1} \subseteq \nu$. Finally, we show that ν is a cardinal in N. Again (2) yields that $M_1 \models \nu \leqq \kappa^+$ since $H \models \nu = \kappa^+$. Assume that ν is no cardinal in M_1. Then there is some $r \subseteq \kappa \times \kappa$ such that $r \in M_1$ and r has order type ν. But then $r \in M \subseteq H$. This is a contradiction since H satisfies enough set theory and $\nu \notin H$. qed (a)

(b) Let $\kappa \in C_N$. So N is the 1-mouse iteration of some mouse $M = J^U_\beta$ at κ. Now $\mathfrak{p}(\kappa) \cap H = \mathfrak{p}(\kappa) \cap N = \mathfrak{p}(\kappa) \cap M$. Hence $\langle H, U \rangle \models$ "U is normal". The same argument shows that $\langle H, U \rangle$ is amenable. qed (b)

(c) follows from the proof of (a). qed

The proof of the following remark is left to the reader.

REMARK. Let $\nu \in S_\kappa$, $N = N(\nu)$, $\bar{N} = \bar{N}(\nu)$, $p = p(\nu)$. Then: $\kappa \in C_N$ iff $n(N) = n(\nu)$, $\kappa \in p$, $h_{\bar{N}}(\kappa \cup (p - \{\kappa\})) \cap (\kappa, \mathrm{at}(N)) = \emptyset$ and $\mathfrak{p}(\kappa) \cap N \subseteq h_{\bar{N}}(\kappa \cup (p - \{\kappa\}))$.

LEMMA 3.14. *Let* $\nu \in S_\kappa$, $\mathrm{cf}(\nu) > \omega$, κ *regular. Assume that there is no countably complete U such that* $\langle H(\nu), U \rangle \models$ "*U is normal*" *and* $\langle H(\nu), U \rangle$ *is amenable. Then there is a club* $C \subseteq S_\kappa \cap \nu$ *such that for all* $\lambda \in C$ *there is no countably complete U such that* $\langle H(\lambda), U \rangle \models$ "*U is normal*" *and* $\langle H(\lambda), U \rangle$ *is amenable.*

PROOF. Set $N = N(\nu)$, $H = H(\nu)$. First we show

(1) $\kappa \notin C_N$.

PROOF. Asume that $\kappa \in C_N$. Then by Lemma 3.13(b) there is some U such that $\langle H, U\rangle$ is amenable and $\langle H, U\rangle \models$ "U is normal". But then U is countably complete since $\mathrm{cf}(\nu) > \omega$ and $\mathrm{cf}(\kappa) > \omega$. This contradicts the assumption.

qed (1)

Now let $\langle C_\eta \mid \eta \in S_\kappa\rangle$ be the sequence which results from the proof of Theorem 3.8 and Lemma 2.2. Set $\bar{C} = C_\nu$. Then $\bar{C}$ is club in ν since $\mathrm{cf}(\nu) > \omega$. The proof of 3.8 and 2.2 shows that:

(2) There is a sequence $\langle \sigma_\lambda \mid \lambda \in \bar{C}\rangle$ such that

(i) $\sigma_\lambda : \bar{N}(\lambda) \to_{\Sigma_0} \bar{N}(\nu)$, $\sigma_\kappa \restriction \lambda = \mathrm{id} \restriction \lambda$, $\sigma_\lambda(p(\lambda)) = p(\nu)$,

(ii) $\lambda, \eta \in \bar{C}, \lambda < \eta \to \mathrm{rng}\, \sigma_\lambda \subseteq \mathrm{rng}\, \sigma_\eta$,

(iii) $\bar{N}(\nu) = \bigcup\{\mathrm{rng}\, \sigma_\lambda \mid \lambda \in \bar{C}\}$.

Now by (1) $\kappa \notin C_N$. Hence by (2) and the remark above there is some $\gamma < \kappa$ such that $\kappa \notin C_{N(\lambda)}$ for all $\lambda \in \bar{C} - \gamma$. Hence by 3.13(a) $C = \bar{C} - \gamma$ satisfies the claim. qed

The next result is a strengthening of the well-known fact that assuming $\neg L^\mu$ we have $(\kappa^+)^K = \kappa^+$ for any weakly compact κ.

LEMMA 3.15. *Assume $\neg L^\mu$. Let $\kappa > \omega$ be regular and $(\kappa^+)^K < \kappa^+$. Then $\square_\kappa^-$ holds.*

PROOF. We may assume that $\kappa \geqq \omega_2$ since $\square_{\omega_1}^-$ holds. Hence by the covering theorem for K we have $\mathrm{cf}((\kappa^+)^K) = \kappa$. Let $\langle C_\nu \mid \nu \in S_\kappa\rangle$ be the natural $\square_\kappa$-sequence in K which is given by Theorem 3.8 and Lemma 2.2. We shall show

(*) There is no unbounded $C \subseteq S_\kappa$ such that $C \cap \nu = C_\nu$ for all $\nu \in S_\kappa$.

It follows easily that $\square_\kappa^-$ holds. To see this just choose some $f: \kappa \to S_\kappa$ which is normal and unbounded. For limit ordinals $\lambda < \kappa$ set $D_\lambda = f^{-1\prime\prime} C_{f(\lambda)}$. Then D_λ is a $\square_\kappa^-$-sequence.

So it remains to prove (*). Assume that C is a counterexample. Then we know again by definition of $\langle C_\nu \mid \nu \in S_\kappa\rangle$ that there is a commutative system $\langle \sigma_{\lambda\nu} \mid \lambda \in C \cap \nu\rangle$ of embeddings $\sigma_{\lambda\nu} : \bar{N}(\lambda) \to_{\Sigma_0} \bar{N}(\nu)$ such that $\sigma_{\lambda\nu} \restriction \lambda = \mathrm{id} \restriction \lambda$ and $\alpha_{\lambda\nu}(p(\lambda)) = p(\nu)$. Set $\tau = (\kappa^+)^K$. Since $\mathrm{cf}(\tau) > \omega$ there is a transitive $\tilde{N}$ and $\sigma_\lambda : \bar{N}(\lambda) \to_{\Sigma_0} \tilde{N}$ for $\lambda \in C$ such that $\langle \tilde{N}, \sigma_\lambda\rangle$ is the direct limit of the system $\langle \sigma_{\lambda\nu}\rangle$. An application of Lemma 10.38 in [4] shows that $\tilde{N} = N^n$, $n = n(\min C)$, for some mouse N. But we also know that $h_{\bar{N}(\lambda)}(\kappa \cup p(\lambda)) = \bar{N}(\lambda)$

for $\lambda \in C$. So we have $h_{\bar{N}}(\kappa \cup p) = \bar{N}$ where $p = \sigma_\lambda(p(\lambda))$. We also have $\tau \subseteq \bar{N}$. Moreover, $\bar{N} \in K$ since $\bar{N} = N^n$ for some mouse N. But then τ is no cardinal in K which is a contradiction. qed

§4. □-principles and ultrafilters

This section contains the main results of this paper. We use our □-principles to get regularity of ultrafilters.

THEOREM 4.1. *Let $\kappa > \omega$ be a successor cardinal or a singular cardinal. Assume that $\square_\kappa^+$ holds. Then every uniform ultrafilter on κ is regular.*

PROOF. If κ is singular let $\square_\kappa^+$ be given by $\square_\kappa^{++}(A)$. In both cases let $\square_\kappa^+$ be given by S_α, C_ν, A_ν, G_α.

Now let U be a uniform ultrafilter on κ. We have to show that U is regular. So if κ is regular, then by Kanamori's theorem we may assume that

$$U \supseteq \{C \subseteq \kappa \mid C \text{ club in } \kappa\}.$$

If κ is singular, then by Lemma 1.3 we may assume that

$$U \supseteq \{C \subseteq \kappa \mid C \cap \tau \text{ club in } \tau \text{ for all } \tau \in A\}.$$

So in both cases by (E0)(d) we have $U \supseteq \{A_\nu \mid \nu \in S_\kappa\}$. If κ is a successor cardinal we may also assume w.l.o.g. that $\sup S_\alpha < \alpha^+$ for all $\alpha < \kappa$. To see this note that (E0)(a)(d) implies that this is satisfied for all $\alpha \geqq \lambda$ if $\kappa = \lambda^+$. But clearly we can assume that $S_\alpha = \varnothing$ for $\alpha < \lambda$ if $\kappa = \lambda^+$.

After these preliminaries we can treat both cases together. So we only use the weaker version of (E2)(b) which is also satisfied in the singular case. For $\alpha \leqq \kappa$ set $\bar{S}_\alpha = \{\nu \in S_\alpha \mid \sup C_\nu = \nu\}$.

CLAIM 1. There are functions $g_\alpha : \bar{S}_\alpha \to \mathrm{On}$ $(\alpha < \kappa)$ such that:
(i) $g_\alpha(\nu) \in C_\nu$,
(ii) $\nu, \tau \in \bar{S}_\alpha, \nu < \tau, \nu \notin C_\tau \to (C_\nu - g_\alpha(\nu)) \cap (C_\tau - g_\alpha(\tau)) = \varnothing$,
(iii) $\lambda, \nu \in \bar{S}_\alpha, \lambda \in C_\nu \to g_\alpha(\lambda) \leqq g_\alpha(\nu)$.

PROOF. Let $\alpha < \kappa$ and $S_\alpha \neq \varnothing$. By recursion we define $g^\nu : \bar{S}_\alpha \cap (\nu + 1) \to \mathrm{On}$ for $\nu \in S_\alpha$ which satisfy the conditions (i)–(iii). After that we can set $g_\alpha = g^\rho$ where $\rho = \max S_\alpha$. Note that $\max S_\alpha$ exists. The initial and successor steps of the recursion are obvious. So let ν be a limit point of S_α. If $\sup C_\nu < \nu$, hence $\mathrm{cf}(\nu) = \omega$, let H be a monotone ω-sequence of successor points in S_α such that

$\sup H = \nu$ and $\min H > \sup C_\nu$. Set $H = \emptyset$ if $\sup C_\nu = \nu$. Now let $\eta \in (\bar{S}_\alpha \cap \nu) - (C \cup H)$. Set

$$\gamma = \sup((C \cup H) \cap \nu) \quad \text{and} \quad \mu = \min((C \cup H) - (\gamma + 1)).$$

Hence $\gamma < \eta < \mu$. Then set $g^\nu(\eta) = \min\{\lambda \in C_\eta \mid \lambda > \gamma \text{ and } \lambda \geqq g^\mu(\eta)\}$. For $\eta \in (C_\nu \cup \{\nu\}) \cap \bar{S}_\alpha$ set $g^\nu(\eta) = 0$. Clearly, g^ν satisfies the conditions (i)–(iii).

qed (Claim 1)

Clearly, we may assume that $A_\nu \subseteq \{\alpha < \kappa \mid \lim(\alpha)\}$. Now let $\nu \in \bar{S}_\kappa$. Then by (E2)(b) we can define a map $f_\nu : A_\nu \to \nu$ by

$$f_\nu(\alpha) = \min\{\rho \in C_\nu \mid G_\alpha(\rho) \geqq g_\alpha(G_\alpha(\nu))\}.$$

CLAIM 2. Let $\gamma < \kappa$, $\lim(\gamma)$. Then

$$Y_\gamma = \{\nu \in S_\kappa \mid \text{otp } C_\nu = \gamma, f_\nu \text{ no } \nu\text{-decomposition of } U\}$$

is not stationary in κ^+.

PROOF. There is a function $h : Y_\gamma \to \kappa^+$, $h(\nu) \in C_\nu$, such that

$$Z_\nu = \{\alpha \in A_\nu \mid f_\nu(\alpha) \leqq h(\nu)\} \in U \qquad \text{for all } \nu \in Y_\gamma.$$

We shall show that $(C_\nu - h(\nu)) \cap (C_\tau - h(\tau)) = \emptyset$ for $\nu, \tau \in Y_\gamma$, $\nu \neq \tau$. It follows immediately that Y_γ is not stationary. So let $\nu, \tau \in Y_\gamma$, $\nu < \tau$. Assume that $\lambda \in (C_\nu - h(\nu)) \cap (C_\tau - h(\tau))$. We shall derive a contradiction. Note that $\nu \notin C_\tau$ since otp $C_\nu =$ otp $C_\tau = \gamma$. Set $\mu = \sup(C_\tau \cap \nu)$, $\rho = \min(C_\tau - \nu)$. Since $Z_\nu \cap Z_\tau \in U$, we can apply (E2)(a) to find some $\alpha \in Z_\nu \cap Z_\tau$ such that $G_\alpha(\mu) < G_\alpha(\nu) < G_\alpha(\rho)$. Set $\bar{\nu} = G_\alpha(\nu)$, $\bar{\tau} = G_\alpha(\tau)$. By (E2)(b) we get that $\bar{\nu} \notin C_{\bar{\tau}}$ and $G_\alpha(\lambda) \in (C_{\bar{\nu}} - g_\alpha(\bar{\nu})) \cap (C_{\bar{\tau}} - g_\alpha(\bar{\tau}))$. This contradicts (ii) of Claim 1.

qed (Claim 2)

For $\gamma < \kappa$, $\lim(\gamma)$, now choose a club $D_\gamma \subseteq \kappa^+$ such that $D_\gamma \cap Y_\gamma = \emptyset$. Set $D = \bigcap\{D_\gamma \mid \gamma < \kappa, \lim(\gamma)\}$. Then D is club in κ^+ and we have

(1) Let $\nu \in \bar{S}_\kappa \cap D$ such that otp $C_\nu < \kappa$. Then f_ν is a ν-decomposition of U.
Now we shall show

(2) Let $\omega < \rho \leqq \kappa$ such that ρ is regular. Then U is (ω, ρ)-regular.
This is sufficient, since we can apply Lemma 1.1 if κ is singular.

So let ρ be as in (2). We shall apply Lemma 1.2. Choose some $\nu \in S_\kappa$ such that $\text{cf}(\nu) = \rho$ and ν is a limit point of D. So there is some $B \subseteq \nu$ such that

(a) B club in ν, $B \subseteq D$, otp $B = \rho$,

(b) $\lambda \in B \to \lambda$ is a limit point of C_ν.

Set $X = A_\nu$. For $\lambda \in B$ define $\bar{f}_\lambda : X \to \lambda$ by

$$\bar{f}_\lambda(\alpha) = \begin{cases} f_\lambda(\alpha) & \text{if } \alpha > \operatorname{otp} C_\lambda, \\ f_\nu(\alpha) & \text{if } \alpha \leqq \operatorname{otp} C_\lambda. \end{cases}$$

Clearly, $\bar{f}_\lambda = f_\lambda \bmod U$. So $\bar{f}_\lambda$ is a λ-decomposition of U since $B \subseteq D$. But (E2)(b) and (iii) of Claim 1 show that $\bar{f}_\lambda \leqq \bar{f}_\mu$ for all $\lambda, \mu \in B$, $\lambda \leqq \mu$. So $\langle \bar{f}_\lambda \mid \lambda \in B \rangle$ shows that U is (ω, ρ)-regular. qed

Theorem 4.2. *Assume $\neg L^\mu$. Let κ be a singular cardinal. Then every uniform ultrafilter on κ is regular.*

Proof. This follows immediately from Theorem 4.1 and Theorem 3.12. qed

Theorem 4.3. *Let $\kappa > \omega$ be regular and assume $(\kappa^+)^K = \kappa^+$. Moreover assume that κ is not measurable in an inner model. Then every uniform ultrafilter on κ is regular.*

Proof. Let U be a uniform ultrafilter on κ. We have to show that U is regular. So by Kanamori's theorem we may assume that $U \supseteq \{C \subseteq \kappa \mid C \text{ club in } \kappa\}$. Let S_α, C_ν, A_ν, G_α be the natural sequence giving $\square_\kappa^+$ in K. For $\nu \in S_\kappa$ define $f_\nu : A_\nu \to \kappa$ by $f_\nu(\alpha) = G_\alpha(\nu)$. Let $g : \kappa \to \mathrm{On}$ be defined by $g(\alpha) = (\alpha^+)^K$. We distinguish two cases.

Case 1. There is some $f <_U g$ such that $f_\nu <_U f$ for all $\nu \in S_\kappa$.

In this case we can argue exactly as in the proof of Theorem 5.1 if we replace A_ν by $A'_\nu = \{\alpha \in A_\nu \mid f_\nu(\alpha) < f(\alpha)\}$ and S_α $(\alpha < \kappa)$ by $S_\alpha \cap (f(\alpha) + 1)$.

Case 2. For all $f <_U g$ there is some $\nu \in S_\kappa$ such that $f \leqq_U f_\nu$.

We now use Lemma 2.3. Hence by taking restrictions we get $\pi_\alpha^\nu : H(G_\alpha(\nu)) \to_{\Sigma_1} H(\nu)$ for $\alpha \in A_\nu$ such that

(1) $\pi_\alpha^\nu \upharpoonright \alpha = \mathrm{id} \upharpoonright \alpha$; $\pi_\alpha^\nu(\alpha) = \kappa$.

(2) Let $\tau \in S_\kappa \cap \nu$, $\tau = \pi_\alpha^\nu(\bar{\tau})$. Then $\bar{\tau} = G_\alpha(\tau)$ and $\pi_\alpha^\tau = \pi_\alpha^\nu \upharpoonright H(G_\alpha(\tau))$.

Moreover, we know by results in §3 that

(3) $\tau \in S_\alpha \cap \nu \to H(\tau) \subseteq H(\nu)$.

(4) ν a limit point of $S_\alpha \to H(\nu) = \bigcup\{H(\tau) \mid \tau \in S_\alpha \cap \nu\}$.

(5) $\alpha > \omega$ a cardinal in $K \to K_{g(\alpha)} \subseteq \bigcup\{H(\nu) \mid \nu \in S_\alpha\}$.

So by the assumption in our case we get

(6) Let $f : A \to V$, $A \in U$, such that $f(\alpha) \in K_{g(\alpha)}$ for all $\alpha \in A$. Then there is some $\nu \in S_\kappa$ such that $\{\alpha \in A \mid f(\alpha) \in H(G_\nu(\alpha))\} \in U$.

For $\nu \in S_\kappa$, $\alpha \in A_\nu$, we set

$$U_\alpha^\nu = \{X \subseteq \alpha \mid X \in H(G_\nu(\alpha)) \text{ and } \alpha \in \pi_\alpha^\nu(X)\}$$

and

$$Z_\nu = \{\alpha \in A_\nu \mid \langle H(G_\nu(\alpha)), U_\alpha^\nu \rangle \text{ is amenable}\}.$$

(7) Let $Z_\nu \in U$. Then $\langle H(\nu), U \cap H(\nu)\rangle$ is amenable and $\langle H(\nu), U \cap H(\nu)\rangle \models$ "$U \cap H(\nu)$ is normal".

PROOF. By (6) there is some $\tau \in S_\kappa$, $\tau > \nu$, such that

$$X \doteq \{\alpha \in Z_\nu \cap A_\tau \mid U_\alpha^\nu \in H(G_\tau(\alpha))\} \in U.$$

We may assume that $H(\tau) \prec K_{\kappa^+}$. Set

$$A = \{\alpha \in A_\tau \mid \nu \in \operatorname{rng} \pi_\alpha^\tau \text{ and } \pi_\alpha^\tau : H(G_\alpha(\tau)) \rightarrow_{\Sigma_\omega} H(\tau)\}.$$

Then A is club in κ. Hence $Z = X \cap A \in U$. For $\alpha \in Z$ set $U^\alpha = \pi_\alpha^\tau(U_\alpha^\nu)$. Then $\langle H(\nu), U^\alpha\rangle$ is amenable and $H(\nu) \models$ "U^α is normal". Moreover, U_α^ν is countably complete in $H(G_\alpha(\tau))$ for all $\alpha \in Z$. To see this let $\langle X_i \mid i < \omega\rangle \in H(G_\alpha(\tau))$ such that $X_i \in U_\alpha^\nu$. Then $\alpha \in \bigcap\{\pi_\alpha^\tau(X_i) \mid i < \omega\}$. Hence U^α is countably complete in $H(\tau)$. But $H(\tau) \prec K_{\kappa^+}$. So U^α is countably complete in K. Applying 3.13(c) in K we get that $U^\alpha = U^\beta$ for all $\alpha, \beta \in Z$. It follows easily that for $\alpha \in Z$ every $Y \in U^\alpha$ countains a nonempty final segment of Z. Hence $U^\alpha = U \cap H(\nu)$ since $Z \in U$.

qed (7)

(8) There is some $\tau \in S_\kappa$ such that $Z_\nu \notin U$ for all $\nu \in S_\kappa - \tau$.

PROOF. Assume not. Then by (7) for cofinally many $\nu \in S_\kappa$ $\langle H(\nu), U \cap H(\nu)\rangle$ is amenable and $\langle H(\nu), U \cap H(\nu)\rangle \models$ "$U \cap H(\nu)$ is normal". But then $\langle K_{\kappa^+}, U \cap K_{\kappa^+}\rangle$ is amenable and $\langle K_{\kappa^+}, U \cap K_{\kappa^+}\rangle \models$ "$U \cap K_{\kappa^+}$ is normal". A standard argument shows that $U \cap K_{\kappa^+}$ is countably complete. So $U \cap L[U]$ is normal in $L[U]$ by Lemma 16.11 in [4]. This contradicts our assumptions.

qed (8)

Now choose τ as in (8). So by (6) we can define recursively a sequence $\langle \gamma(\delta) \mid \delta < \kappa\rangle$ with the properties

(9) (a) $\langle \gamma(\delta) \mid \delta < \kappa\rangle$ is normal, $\gamma(0) = \tau$, $\gamma(\delta) \in S_\kappa$;

(b) $X_\delta = \{\alpha \in A_{\gamma(\delta)} \cap A_{\gamma(\delta+1)} \mid U_\alpha^{\gamma(\delta)} \in H(G_\alpha(\gamma(\delta+1)))\} \in U$ for all $\delta < \kappa$.

Now set $\gamma = \sup\{\gamma(\delta) \mid \delta < \kappa\}$.

(10) There are some $Y \subseteq A_\gamma$, $Y \in U$, and a club $C \subsetneq \{\gamma(\delta) \mid \delta < \kappa\}$ with the property:

Let $\alpha \in Y$, $\eta \in (C \cap \operatorname{rng} \pi_\alpha^\gamma) \cup \{\gamma\}$.

Then $\langle H(G_\alpha(\eta)), U_\alpha^\eta \rangle$ is not amenable.

PROOF. By (6) we can choose a $\nu \in S_\kappa$, $\nu > \gamma$, such that $H(\nu) \prec K_{\kappa^+}$ and $\bar{Y} = \{\alpha \in A_\nu \mid \gamma \in \operatorname{rng} \pi_\alpha^\nu,\ U_\alpha^\gamma \in H(G_\alpha(\nu))\} \in U$. We distinguish two cases.

Case A. There is some $D \in K$ such that $\langle H(\gamma), D \rangle$ is amenable, $\langle H(\gamma), D \rangle \models$ "D normal" and D is countably complete in K.

Then there is a club $C \subseteq \{\gamma(\delta) \mid \delta < \kappa\}$ such that $\langle H(\eta), D \cap H(\eta) \rangle$ is amenable for all $\eta \in C$. Let $\rho = \min C$. Then set:

$$Y = (\kappa - Z_\rho) \cap \bar{Y} \cap \{\alpha \in A_\gamma \mid \rho \in \operatorname{rng} \pi_\alpha^\nu\}.$$

Hence $Y \in U$. Now let $\alpha \in Y$, $\eta \in (C \cap \operatorname{rng} \pi_\alpha^\gamma) \cup \{\gamma\}$. Assume that $\langle H(G_\alpha(\eta)), U_\alpha^\eta \rangle$ were amenable. By uniqueness (see Lemma 3.13(c)) we get $\pi_\alpha^\nu(U_\alpha^\eta) = D \cap H(\eta)$. But $D \cap H(\rho)$ is amenable and $\pi_\alpha^\nu(G_\alpha(\rho)) = \rho$. So $\langle H(G_\alpha(\rho)), U_\alpha^\eta \cap G_\alpha(\rho) \rangle$ is amenable. But $U_\alpha^\eta \cap G_\alpha(\rho) = U_\alpha^\rho$. Hence $\alpha \in Z_\rho$ which is a contradiction.

Case B. Case A does not hold.

So by Lemma 3.14 there is a club $C \subseteq \kappa$ such that for all $\eta \in C$ there is no $D \in K$ such that $\langle H(\eta), D \rangle$ is amenable. $\langle H(\eta), D \rangle \models$ "D normal" and D is countably complete in K. Now set $Y = \bar{Y}$. Let $\alpha \in Y$, $\eta \in (C \cap \operatorname{rng} \pi_\alpha^\gamma) \cup \{\gamma\}$. Then $\langle H(G_\alpha(\eta)), U_\alpha^\eta \rangle$ cannot be amenable, since otherwise $\pi_\alpha^\nu(U_\alpha^\eta)$ would show that $\nu \notin C$. qed (10)

Now let Y, C be as in (10). Set

$$A = \{\alpha \in A_\gamma \mid C \cap \operatorname{rng} \pi_\alpha^\gamma \text{ is an initial segment of } C\}.$$

Then A is club in κ, hence $X = Y \cap A \in U$. Let C^* be the set of limit points of C in γ. For $\eta \in C^*$ we can define $f_\eta : X \to \eta$ by

$$f_\eta(\alpha) = \begin{cases} \min\{\rho \in C \cap \eta \mid U_\alpha^\rho \notin H(G_\alpha(\eta))\} & \text{if } \eta \in \operatorname{rng} \pi_\alpha^\gamma, \\ \min\{\rho \in C \cap \operatorname{rng} \pi_\alpha^\gamma \mid U_\alpha^\rho \notin H(G_\alpha(\gamma))\} & \text{otherwise.} \end{cases}$$

Obviously, $f_\eta \leqq f_\rho$ for all $\eta, \rho \in C^*$, $\eta \leqq \rho$. But (9) implies that f_η is an η-decomposition of U for every $\eta \in C^*$. So by Lemma 1.2 $\langle f_\eta \mid \eta \in C^* \rangle$ shows that U is regular. qed

As a corollary to the proof we get:

COROLLARY 4.4. *Let $\kappa > \omega$ be regular and $(\kappa^+)^K = \kappa^+$. Let U be a non-*

regular ultrafilter on κ *such that* $U \supseteq \{C \subseteq \kappa \mid C$ *club in* $\kappa\}$. *Then* $U \cap L[U]$ *is normal in* $L[U]$.

Finally, we show:

THEOREM 4.5. *Assume* $\neg L^{\mu}$. *Let* $\kappa > \omega$ *be a cardinal and* U *a uniform ultrafilter on* κ. *Then* U *is* (ω, ρ)*-regular for all* $\rho < \kappa$.

PROOF. By Theorems 4.2, 4.3 we only have to treat the case that κ is regular and $(\kappa^+)^K < \kappa^+$. But then $\square_\kappa^-$ holds by Lemma 3.15. Then the claim follows from Theorem 1.4. qed

We finish this paper with a few remarks. The proofs show that for regular $\kappa < \omega$, $\square_\kappa^+$ holds if $V = L[A]$ for some $A \subseteq \kappa$. So it holds if κ^+ is not inaccessible in K. This already shows that to get a nonregular ultrafilter on a successor cardinal you need the consistency of an incaccessible cardinal. Using additional known ideas one can show that this is true for arbitrary regular cardinals.

Moreover, slight variations of our proofs give stronger results for filters. For example, if $\kappa > \omega$ is regular, then $\square_\kappa^+$ implies that every κ^+-saturated filter $\mathscr{F}$ on κ containing all club subsets of κ is regular. Here regularity for filters should be defined exactly as for ultrafilters, i.e. using sets of $\mathscr{F}$-measure one. There is also a suitable version of Kanamori's thoerem for uniform κ^+-saturated filters on κ. The situation is slightly different in Theorem 4.3 because the distinction of the two cases uses the ultrafilter property. Moreover, Theorem 1.4 only holds for κ-saturated uniform filters on κ. But we have a different proof of Theorem 4.5 which does not use Theorem 1.4. Details concerning these remarks will appear in a later paper.

REFERENCES

1. M. Benda and J. Ketonen, *Regularity of ultrafilters*, Isr. J. Math. **17** (1974), 231–240.

2. C. C. Chang, *Extensions of combinatorial principles of Kurepa and Prikry*, in *Theory of Sets and Topology* (G. Asser, ed.), Springer-Verlag, Berlin, 1972, pp. 49–57.

3. C. C. Chang and H. J. Keisler, *Model Theory*, North-Holland, Amsterdam, 1973.

4. A. J. Dodd, *The Core Model*, London Math. Soc. Lecture Note Series 61, Cambridge University Press, 1982.

5. A. J. Dodd and R. B. Jensen, *The core model*, Ann. Math. Logic **20** (1981), 43–75.

6. A. J. Dodd and R. B. Jensen, *The covering lemma for K*, Ann. Math. Logic **22** (1982), 1–30.

7. H.-D. Donder, R. B. Jensen and B. J. Koppelberg, *Some applications of the core model*, in *Set Theory and Model Theory, Proc. Bonn 1979*, Lecture Notes in Math. **872**, Springer-Verlag, Berlin, 1981, pp. 55–97.

8. R. B. Jensen, *Some combinatorial principles of L and V*, unpublished manuscript.

9. R. B. Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic **4** (1972), 229–308.

10. A. Kanamori, *Weakly normal filters and irregular ultrafilters*, Trans. Am. Math. Soc. **220** (1976), 393–399.

11. A. Kanamori, W. Reinhardt and R. Solovay, *Strong axioms of infinity and elementary embeddings*, Ann. Math. Logic **13** (1978), 73–116.

12. J. Ketonen, *Nonregular ultrafilters and large cardinals*, Trans. Am. Math. Soc. **224** (1976), 61–73.

13. M. Magidor, *On the existence of nonregular ultrafilters and the cardinality of ultrapowers*, Trans. Am. Math. Soc. **249** (1979), 97–111.

14. K. Prikry, *Changing measurable into accessible cardinals*, Diss. Math. **68** (1970), 5–52.

15. K. Prikry, *On a problem of Gillman and Keisler*, Ann. Math. Logic **2** (1970), 179–187.

16. K. Prikry, *On descendingly complete ultrafilters*, Cambridge Summer School in Mathematical Logic, Lecture Notes in Math. **337**, Springer-Verlag, Berlin, 1973, pp. 459–488.

17. P. Welch, *Combinatorial Principles in the core model*, D. Phil. thesis, Oxford, 1979.